Gompertz in early life
My on-off-on relationship with Vaupelian modeling

Jonas Schöley

@jschoeley.com
0000-0002-3340-8518
schoeley@demogr.mpg.de

MAX PLANCK INSTITUTE
FOR DEMOGRAPHIC RESEARCH

Valen, Leigh Van. 1975. "Life, Death, and Energy of a Tree."

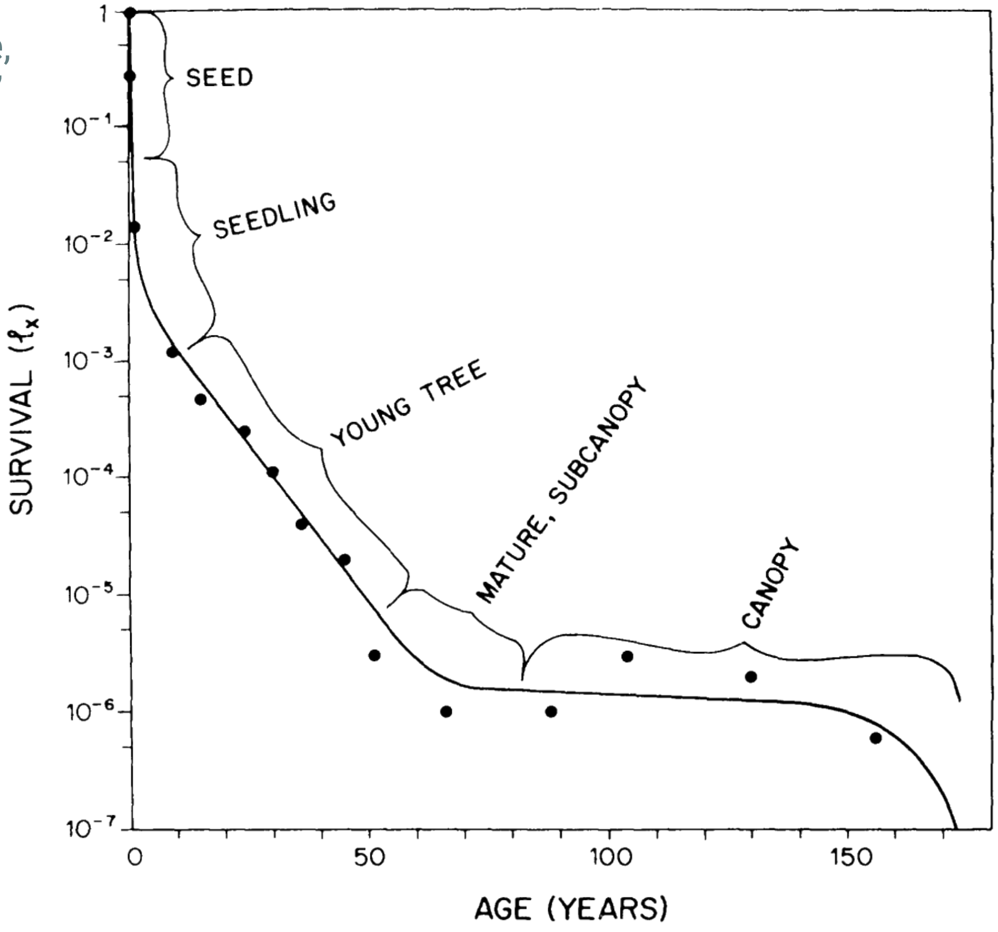
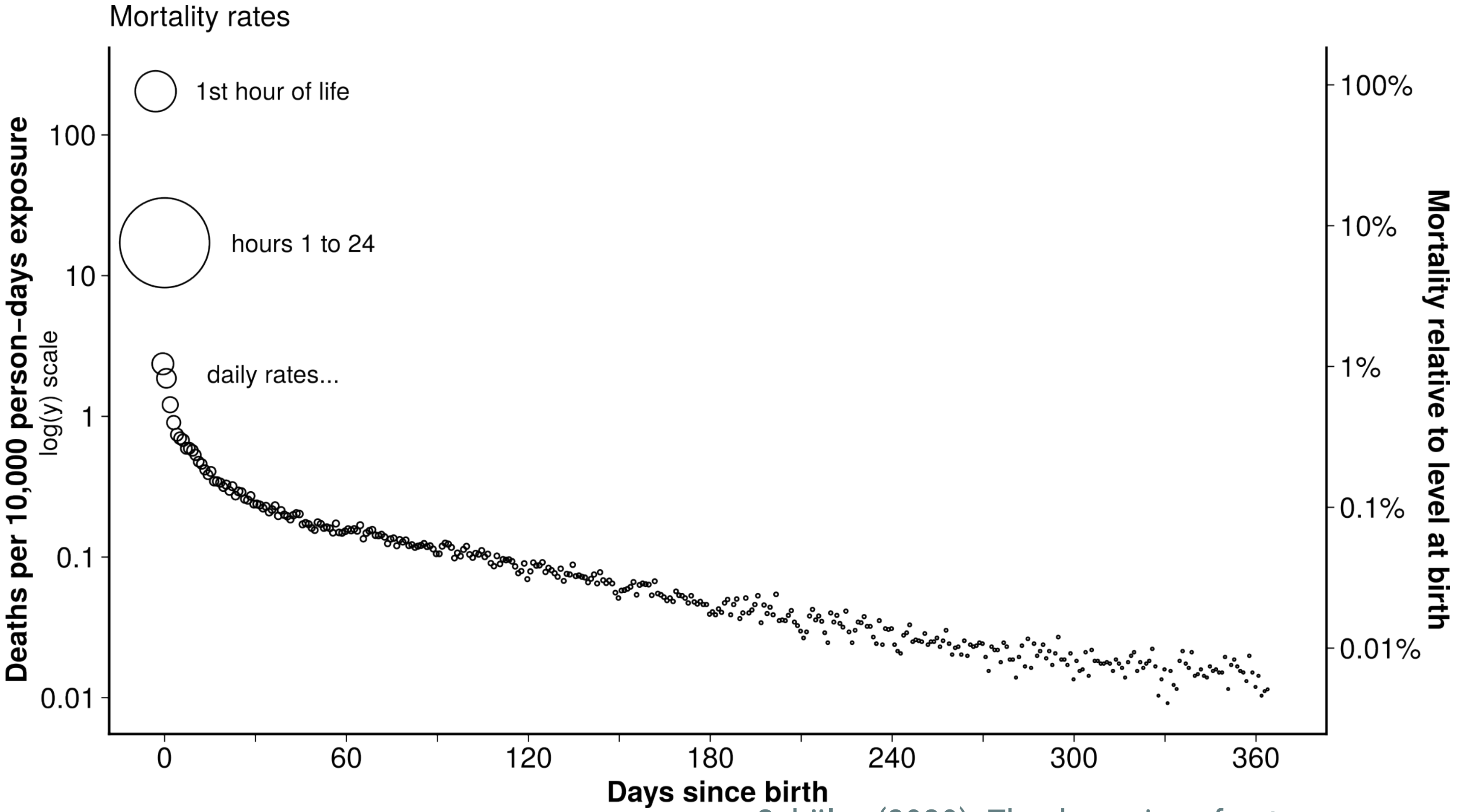


FIGURE 1. Survivorship curve for *Euterpe globosa*, over 7 orders of magnitude. All values of l_x except that at 9 years are completely independent of each other, thus providing an internal check on accuracy.

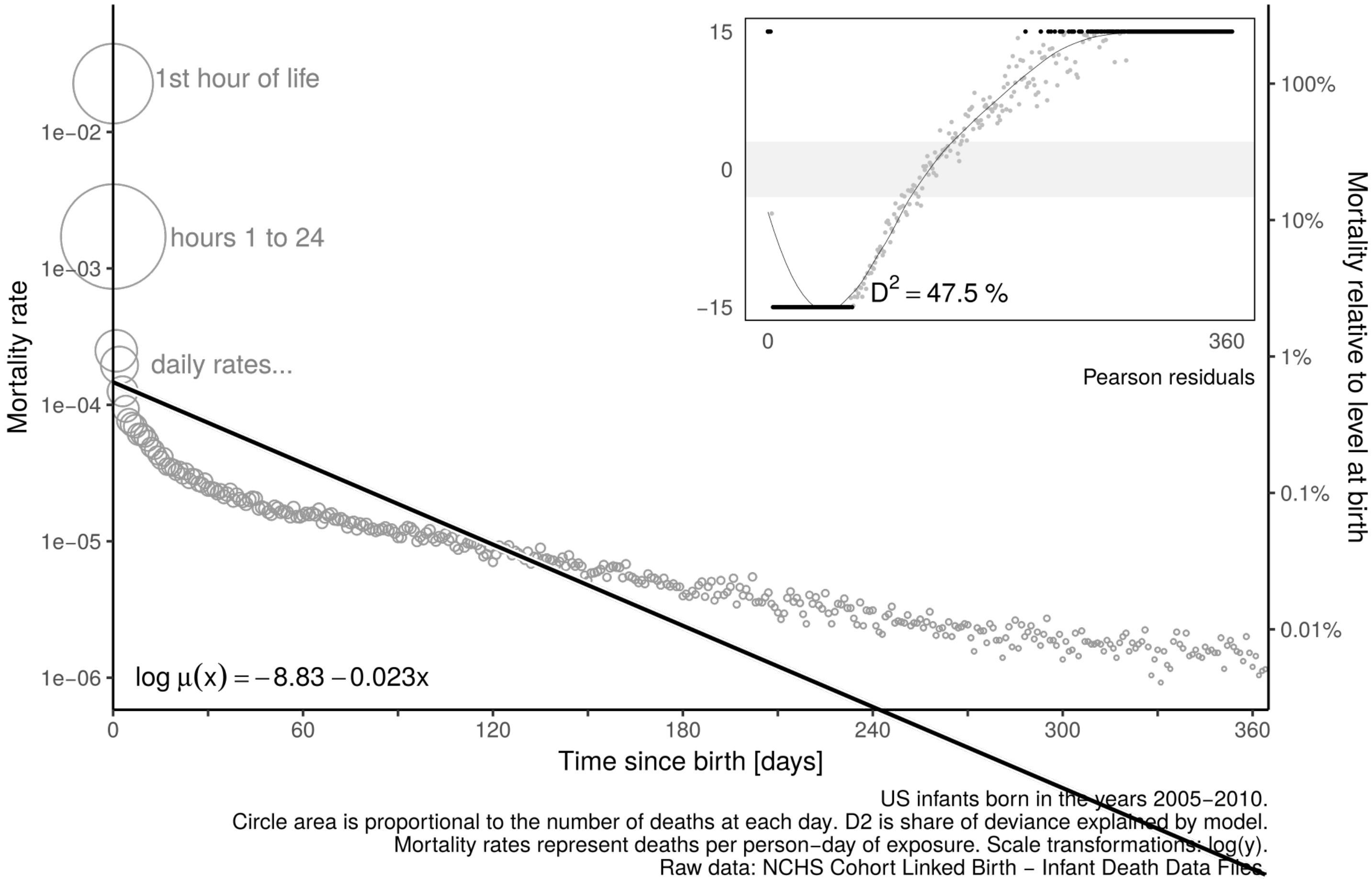
Laws of mortality

Laws of mortality a single parameter!



Schöley (2020). The dynamics of ontogenescence.

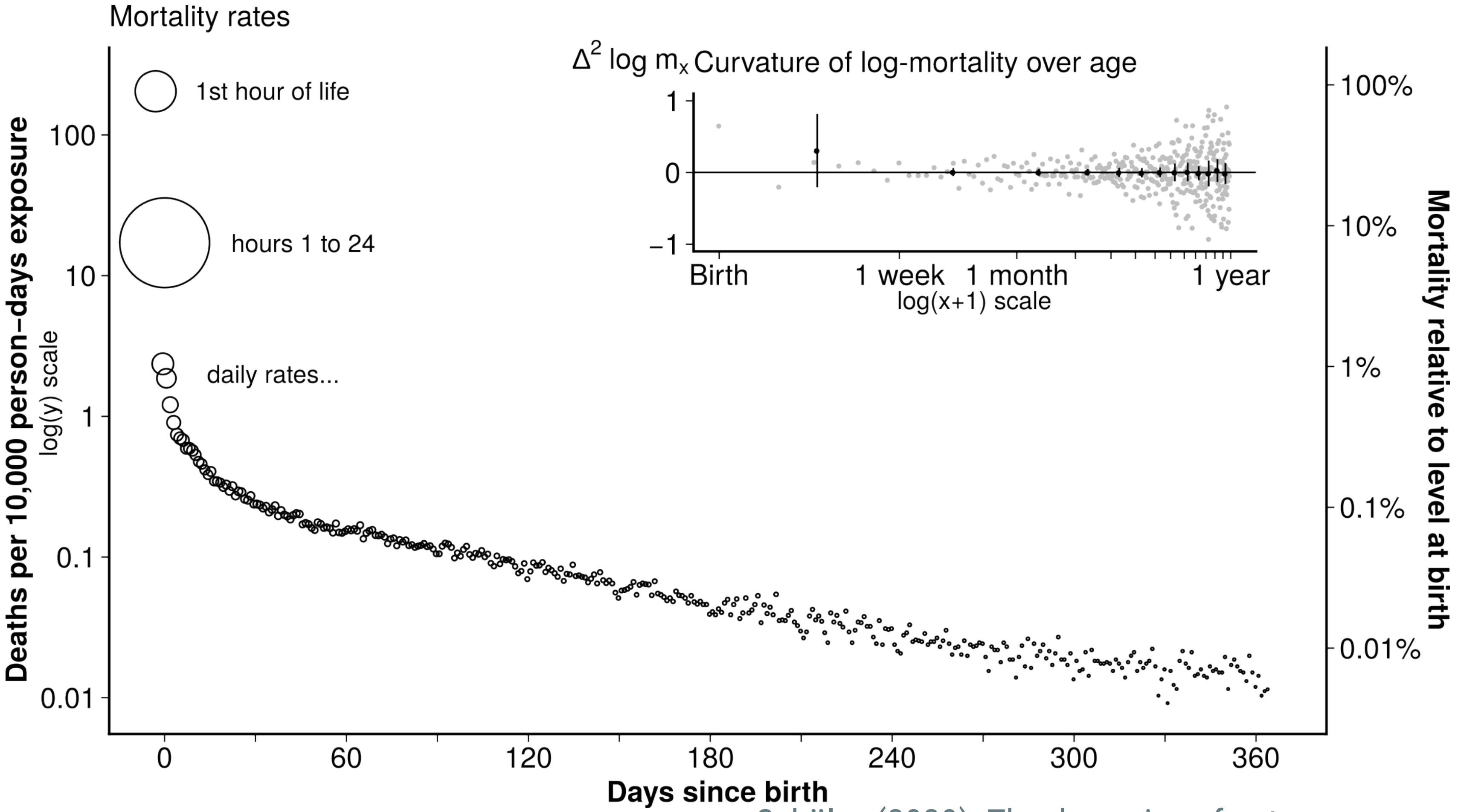
Laws of mortality a single parameter!



US infants born in the years 2005−2010.
Circle area is proportional to the number of deaths at each day. D2 is share of deviance explained by model.
Mortality rates represent deaths per person−day of exposure. Scale transformations: log(y).
Raw data: NCHS Cohort Linked Birth − Infant Death Data Files.

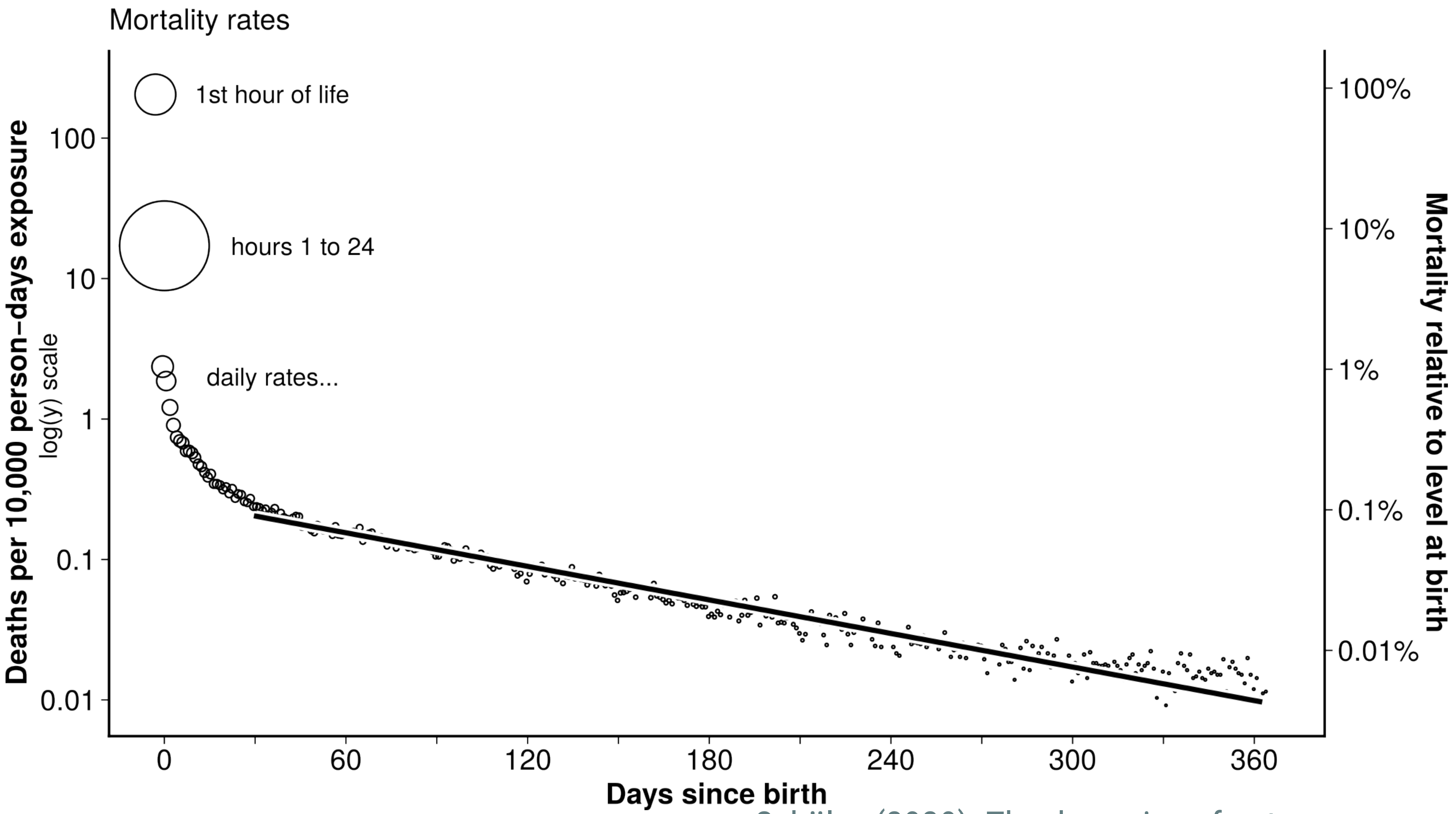
Schöley (2020). The dynamics of ontogenescence.

Laws of mortality a single parameter!



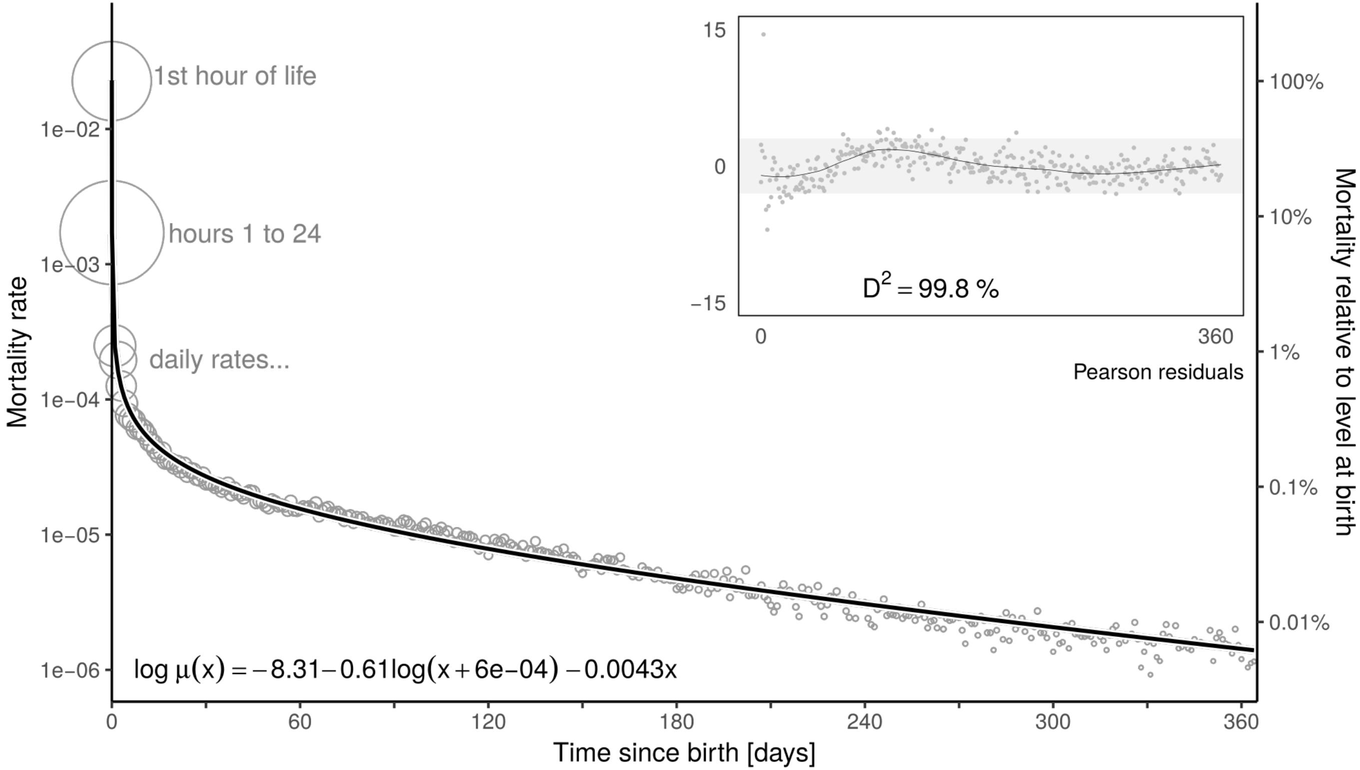
Schöley (2020). The dynamics of ontogenescence.

Laws of mortality a single parameter!



Schöley (2020). The dynamics of ontogenescence.

Laws of mortality

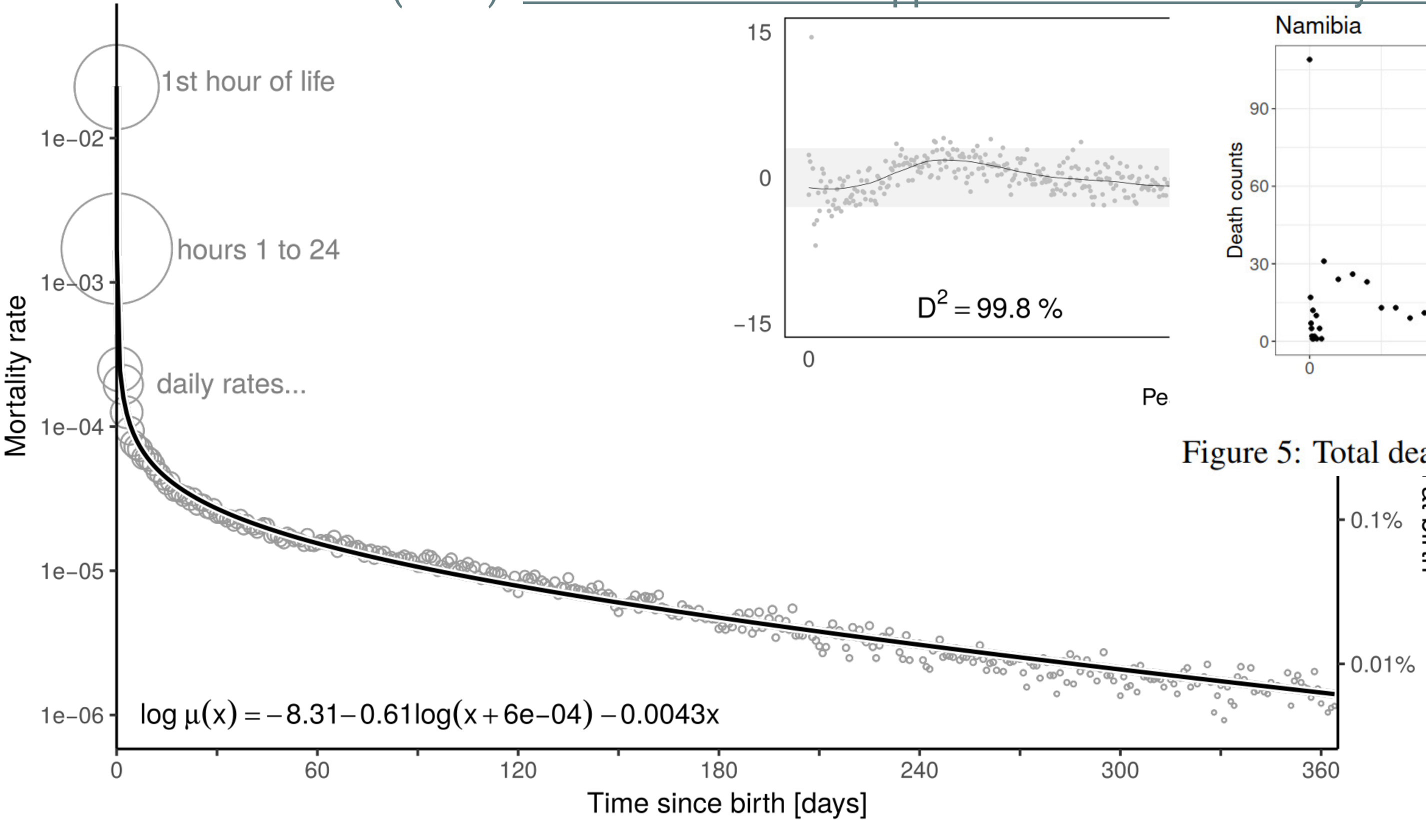


US infants born in the years 2005−2010.
Circle area is proportional to the number of deaths at each day. D2 is share of deviance explained by model.
Mortality rates represent deaths per person−day of exposure. Scale transformations: log(y).
Raw data: NCHS Cohort Linked Birth − Infant Death Data Files.

Schöley (2020). The dynamics of ontogenescence.

Laws of mortality graduation

Okonek et al. (2024). A Pseudo-likelihood Approach to Under-5 MortalityEstimation.



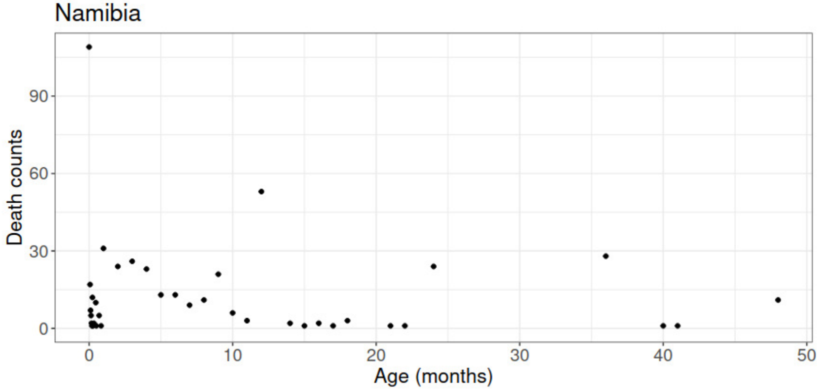
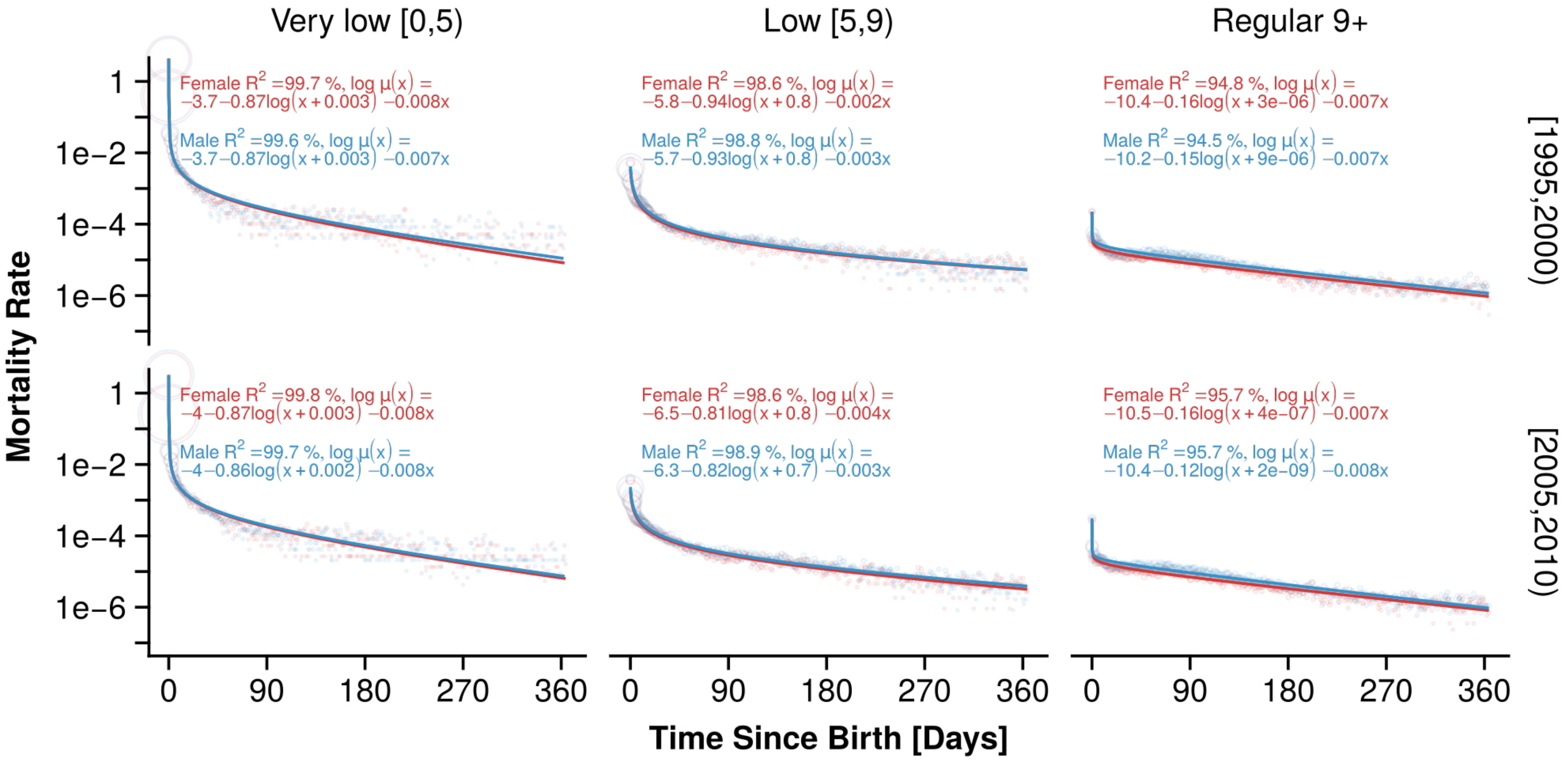


Figure 5: Total death counts at each age group within DHS surveys.

US infants born in the years 2005−2010.
Circle area is proportional to the number of deaths at each day. D2 is share of deviance explained by model.
Mortality rates represent deaths per person−day of exposure. Scale transformations: log(y).
Raw data: NCHS Cohort Linked Birth − Infant Death Data Files.

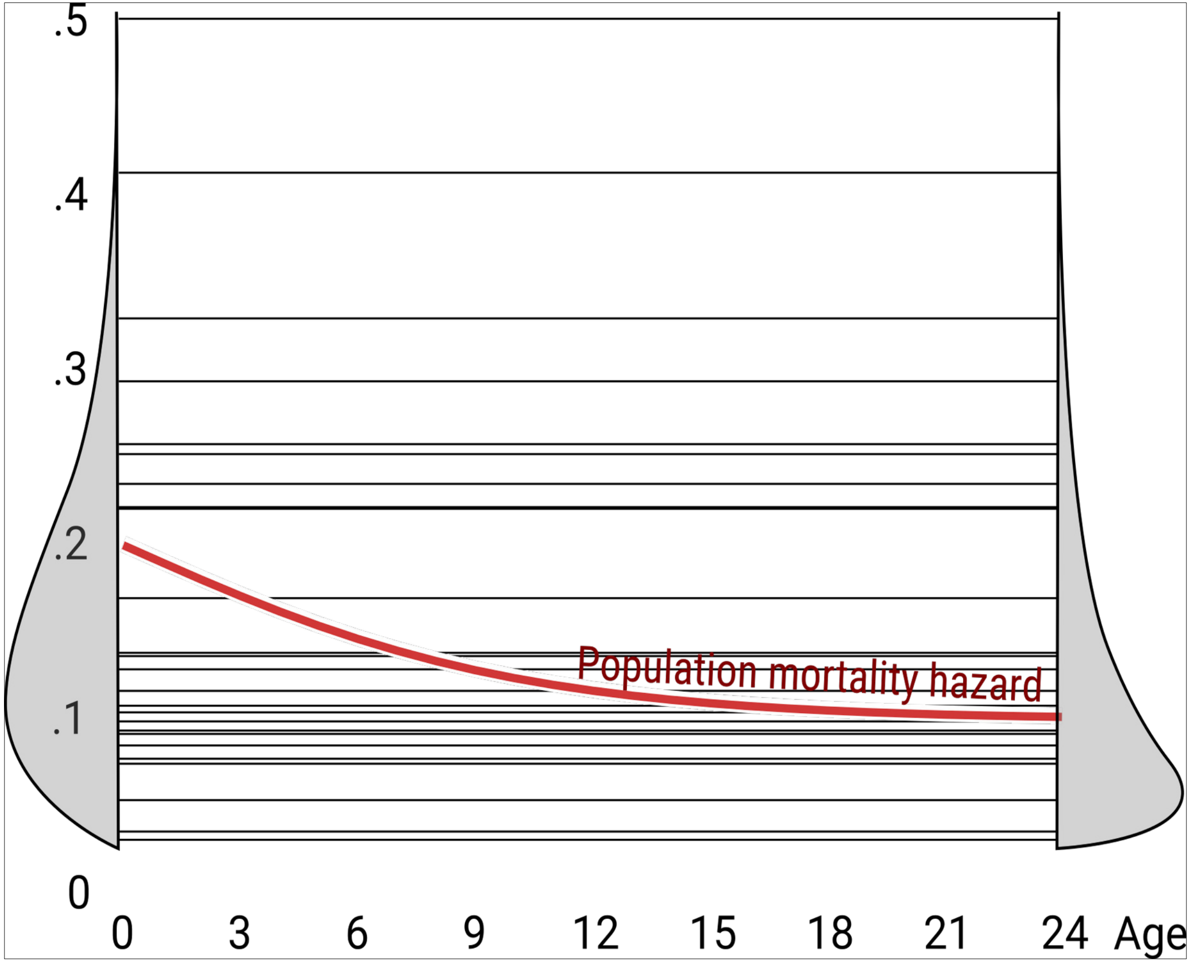
Laws of mortality compression



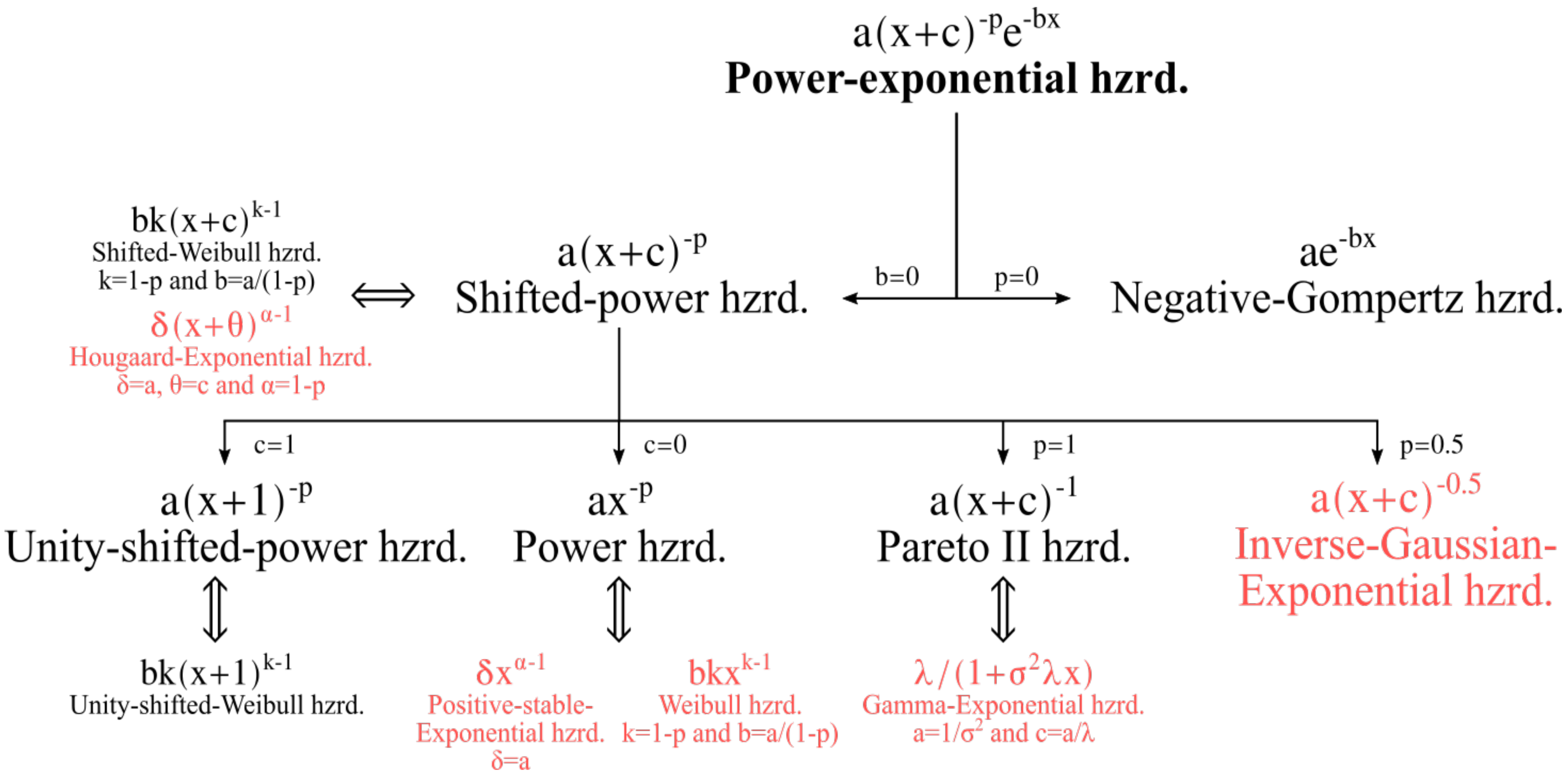
Schöley (2020). The dynamics of ontogenescence.

Mechanisms of mortality

Mechanisms of mortality characterization of power-exponential law



Mechanisms of mortality characterization of power-exponential law



Mechanisms of mortality characterization of power-exponential law

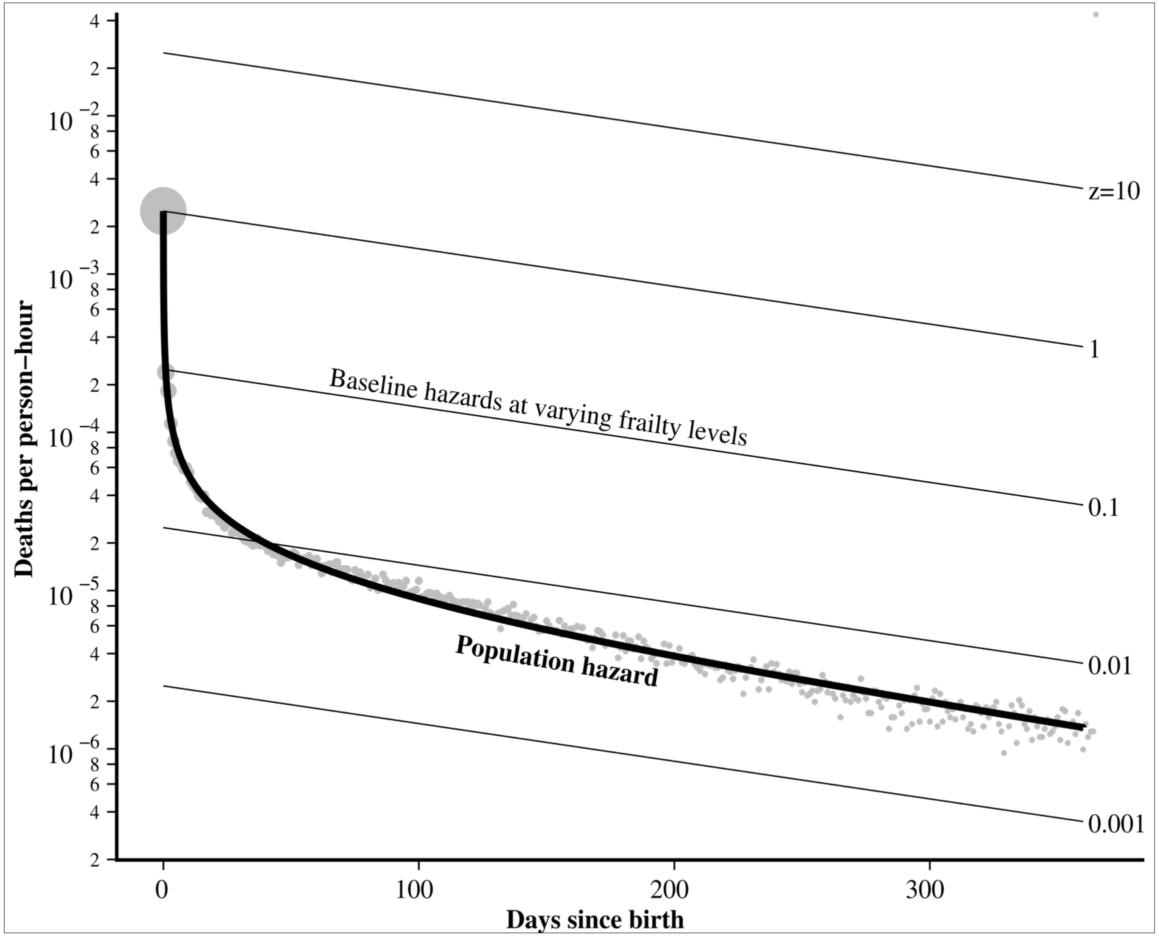
Hougaaard (1986). Survival models for heterogeneous populations derived from stable distributions.

Power-exponential hazard ≈ Hougaard-Gompertz frailty model

$$a(x+c)^{-p}e^{-bx} \approx a[(1-\exp(-bx))/b + c]^{-p}e^{-bx}$$

for *x* close to 0

Mechanisms of mortality characterization of power-exponential law



Hougaard (1986) describes a class of multiplicative frailty models which exhibit a shifted-power-law decline in average frailty over age. Frailty is assumed to be distributed according to a three parameter extension of the stable distributions $f_Z(\alpha, \delta, \theta)$, a density without closed-form representation, but with closed-form Laplace transform

$$L\{f_Z(\alpha, \delta, \theta)\}(s) = \mathrm{E}[e^{-sZ}] = \exp\left[-\frac{\delta}{\alpha}\left[(\theta+s)^\alpha - \theta^\alpha\right]\right].$$

Substituting the cumulative baseline hazard $H_0(x) = \int_0^{x=s} h_0(s)\,\mathrm{d}s$ for argument s one recovers the population survival function

$$\overline{S}(x) = \mathrm{E}\left[e^{-H_0(x)Z}\right] = \exp\left[-\frac{\delta}{\alpha}\left[(\theta + H_0(x))^\alpha - \theta^\alpha\right]\right]$$

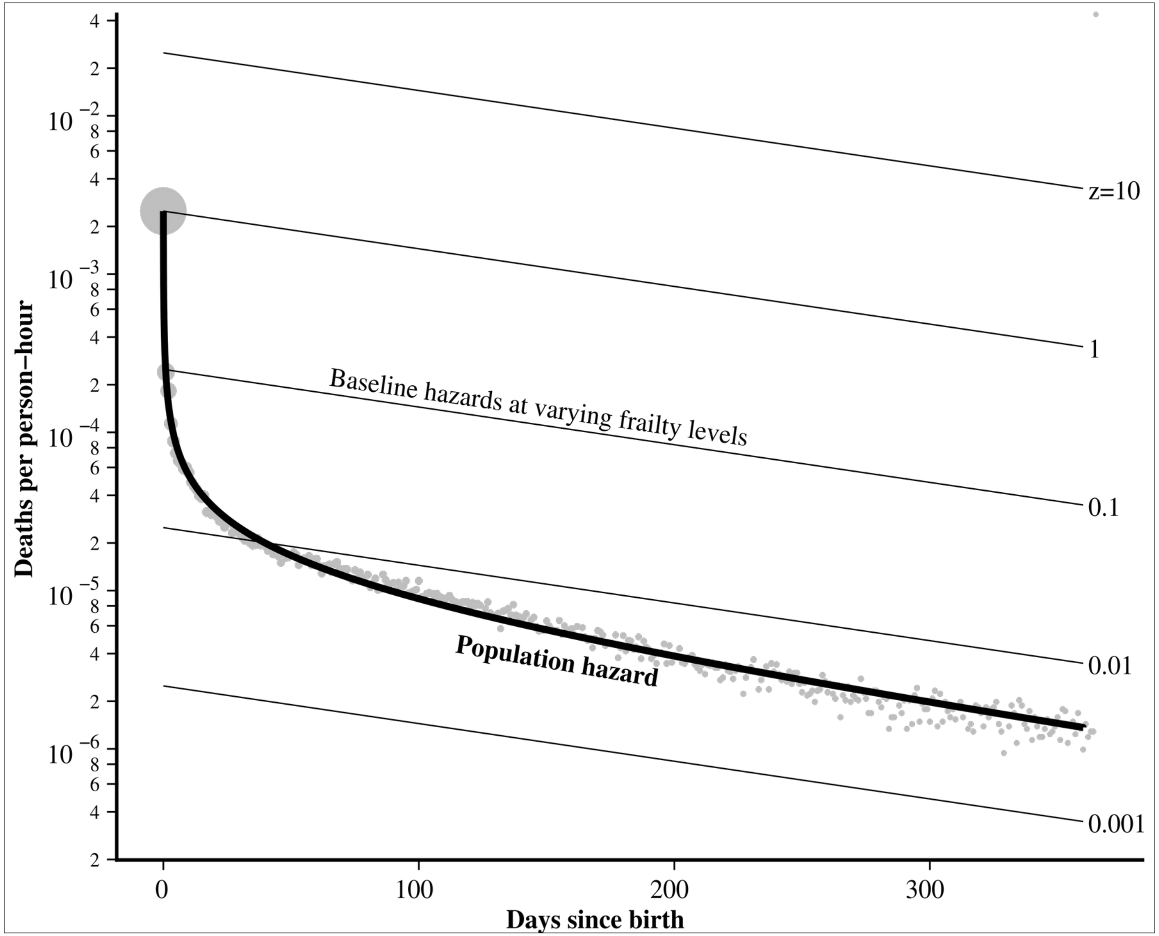
and corresponding population hazard

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}\log \overline{S}(x) &= \overline{h}(x) \\ &= \mathrm{E}[Z|x]h_0(x) \\ &= \delta(\theta + H_0(x))^{\alpha-1}h_0(x). \end{aligned}$$

Assuming a negative-Gompertz baseline hazard $h_0 = \exp(-bx)$ and substituting $a = \delta$, $c = \theta$ and $p = -\alpha + 1$ into $\overline{h}(x)$ yields the Hougaard-Negative-Gompertz hazard

$$h_{\mathrm{HG}}(x) = a\left(\frac{1-e^{-bx}}{b} + c\right)^{-p} e^{-bx}.$$

Mechanisms of mortality characterization of power-exponential law



7.1 A shock-recovery process

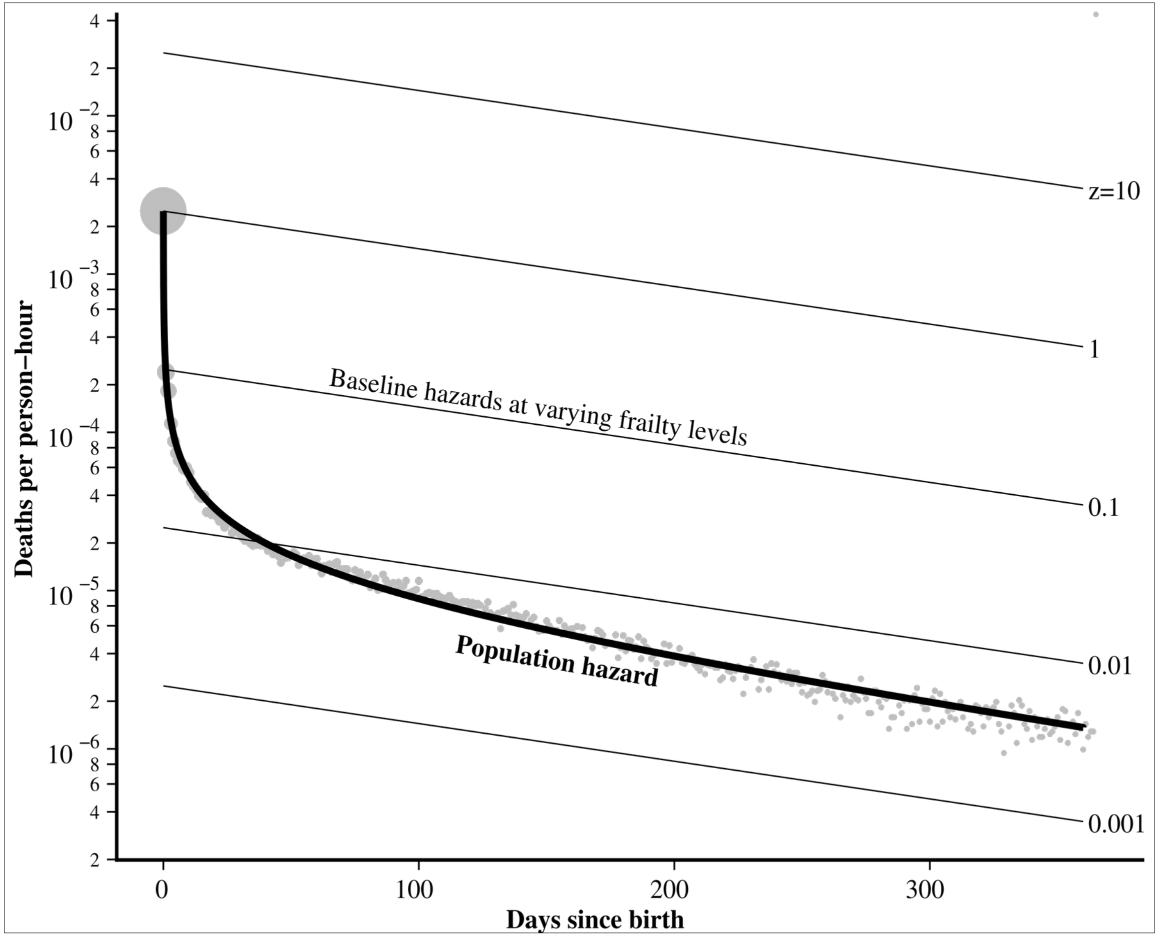
The power-exponential product of the population hazard h_{PE} can be interpreted as a *non-homogeneous split Poisson process*, where *shocks* to an infant's health arrive with rate $\lambda(x)$ per unit person-time, each shock resulting in infant death with probability $p(x)$. Shock models in the context of human mortality have for example been studied by Strehler and Mildvan (1960), Finkelstein (2005), Cha and Finkelstein (2016). A similar explanation for the age-trajectory of preadolecent mortality across species has been put forward by Levitis (2011) under the name "transitional timing hypothesis" stating that "transcriptional, developmental and environmental transitions are dangerous, and these are concentrated early in life."

Let $N(x)$ be the number of infants alive at age x and let $\mathrm{E}[M]$ be the expected value of a Poisson distributed random variable M with rate parameter $\int_x^{x+n} \lambda(x)N(x)$ representing the total number of *health-shocks* the population of infants is expected to experience over age interval $[x, x+n)$. If each shock leads to death with probability $p(x)$ then the number of deaths D over age interval $[x, x+n)$ follows a Poisson distribution with expected value

$$\mathrm{E}[{}_nD_x] = \int_x^{x+n} \lambda(x)p(x)N(x)\,\mathrm{d}x,$$

see Prékopa (1958) for a proof. The hazard of death experienced by survivors N at time x is $h(x) = \lambda(x)p(x)$. If the rate of shocks $\lambda(x)$ varies over time according to a flexibly-shifted-power hazard and if the probability of a shock leading to death $p(x)$ is exponentially declining the power-exponential hazard is recovered. Note that neither the rate of shocks nor the probability of death following a shock are identified as this would require inferring the model $h(x) = a_1(x+c)^{-p} \times a_2 e^{-bx}$ from the fit $h(x) = a(x+c)^{-p}e^{-bx}$ – a problem with infinitely many solutions. However, the power and the exponential rate pa-

Mechanisms of mortality characterization of power-exponential law



Mortality selection

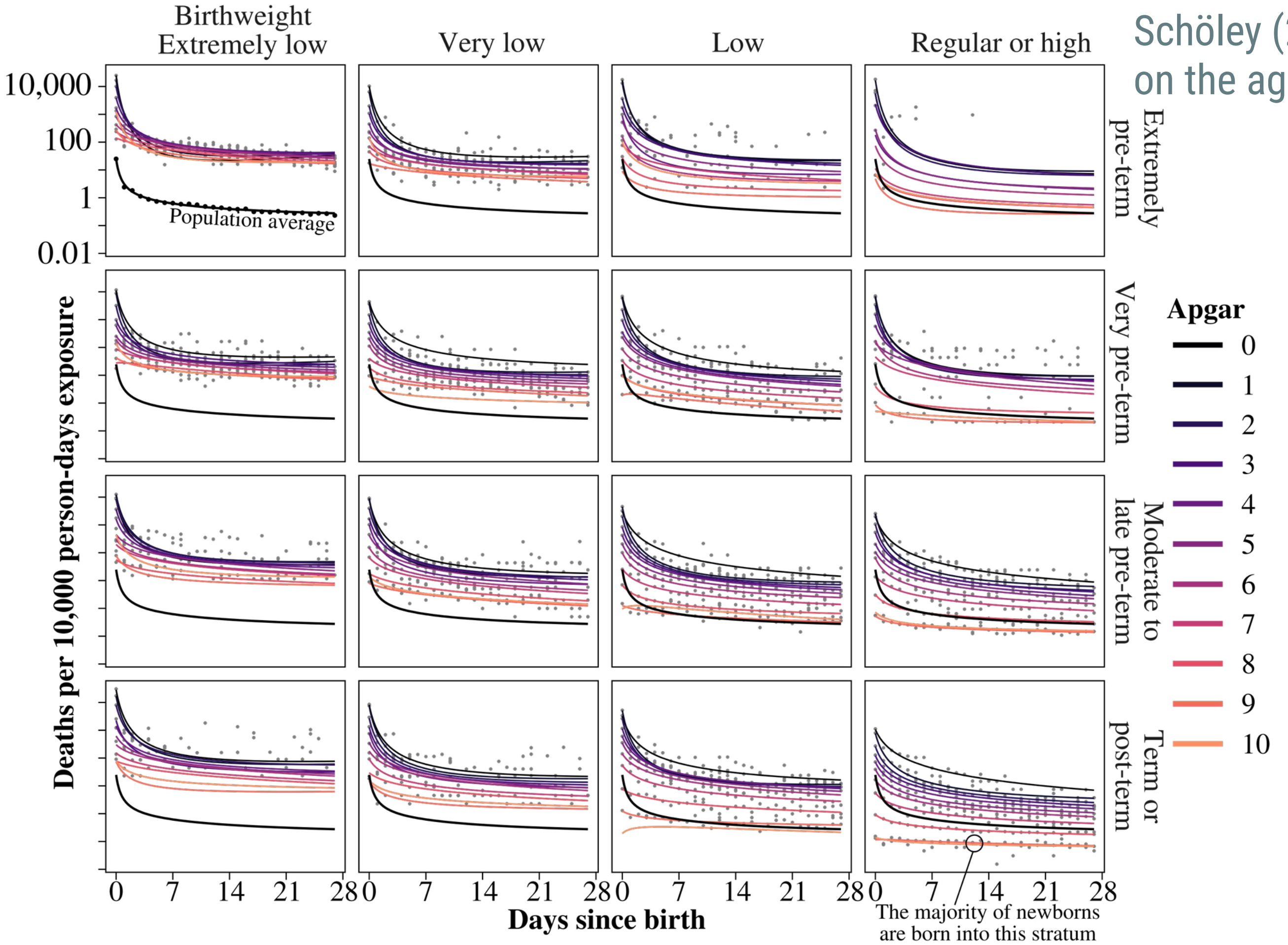
$$a(x+c)^{-p} \times \exp(-bx) = E[Z|x]\mu_0(x)$$

Shock-recovery

$$a(x+c)^{-p} \times \exp(-bx) = \lambda(x)p(x)$$

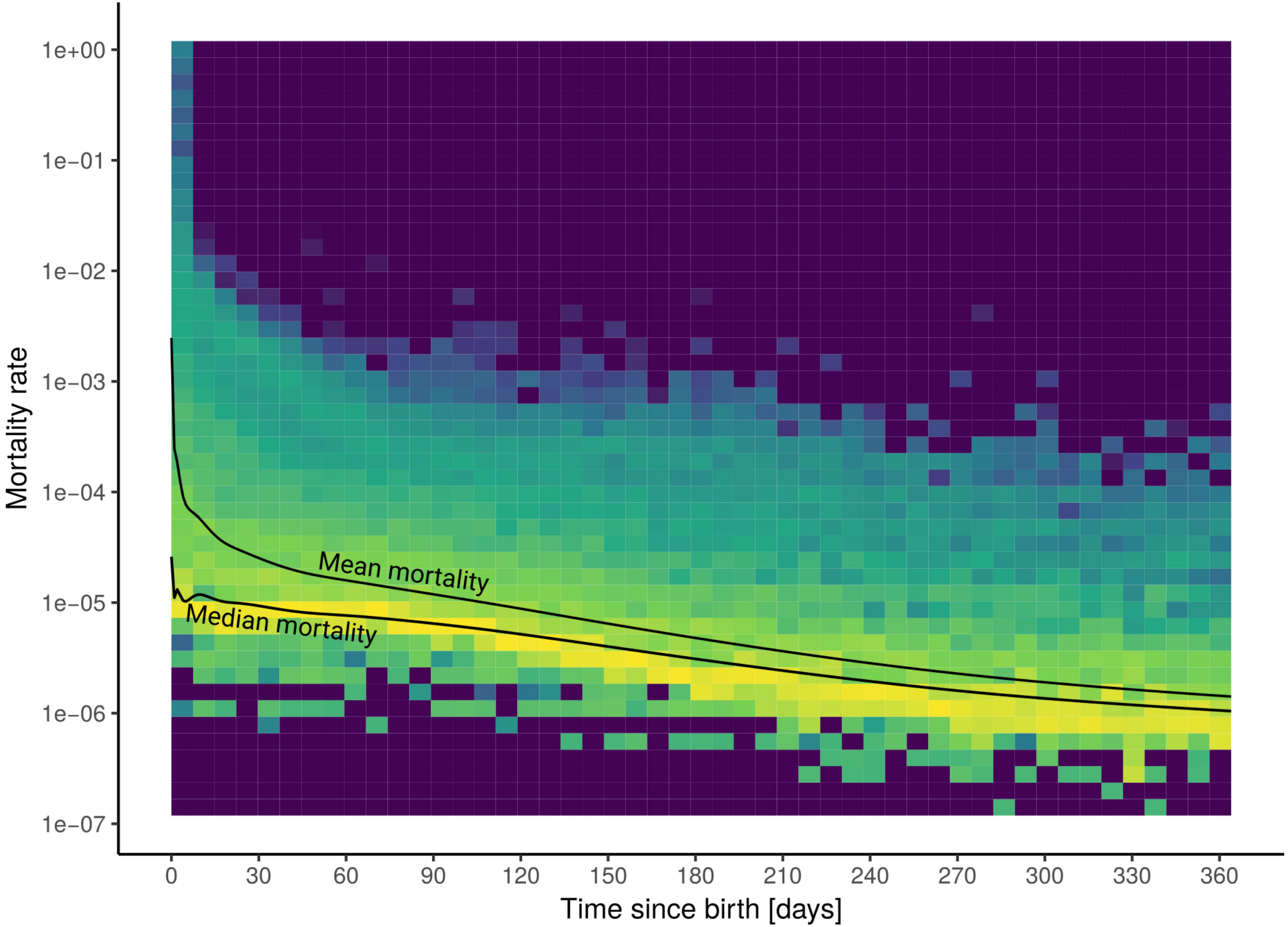
Data over models

Data over models most of us were born Gompertz

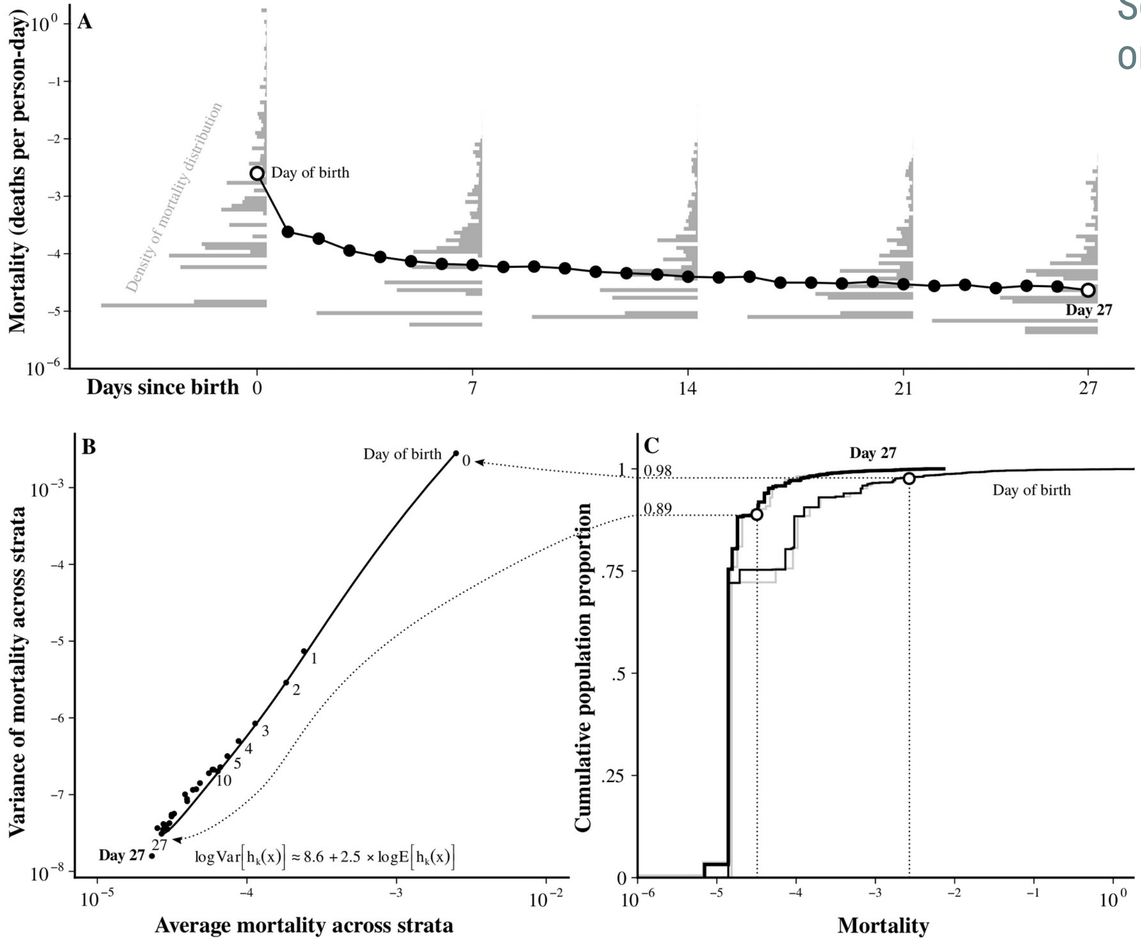


Schöley (2025). The impact of population heterogeneity on the age trajectory of neonatal mortality.

Data over models unveiling hidden heterogeneity



Data over models unveiling hidden heterogeneity



Schöley (2025). The impact of population heterogeneity on the age trajectory of neonatal mortality.

Data over models testing the selection hypothesis

Decomposing **change** in **mean-mode mortality** ratio **over age**

$$\Delta r_j = \underbrace{\sum_k \frac{p_{jk} + p_{j+1,k}}{2} \Delta \frac{m_{jk}}{\mathcal{M}_j}}_{\text{Direct change } \Delta r_j^D} + \underbrace{\sum_k \frac{\frac{m_{jk}}{\mathcal{M}_j} + \frac{m_{j+1,k}}{\mathcal{M}_{j+1}}}{2} \Delta p_{jk}}_{\text{Compositional change } \Delta r_j^C}$$

$\mathcal{M}_j = m_{j,k=r}$ with r such that $p_{j,k=r} = \max(p_{j1}, \ldots, p_{jK})$

$$\frac{\bar{m}_j}{\mathcal{M}_j} = \sum_k p_{jk} \frac{m_{jk}}{\mathcal{M}_j}$$

Decomposing **change** in **mortality variance over age**

$$\Delta v_j = \underbrace{\sum_k \frac{p_{jk} + p_{j+1,k}}{2} \Delta s_{jk}}_{\text{Direct change } \Delta v_j^D} + \underbrace{\sum_k \frac{s_{jk} + s_{j+1,k}}{2} \Delta p_{jk}}_{\text{Compositional change } \Delta v_j^C}$$

$v_j(x) = \sum_k p_{jk} s_{jk}$, with $s_{jk} = (m_{jk} - \bar{m}_j)^2$

Decomposing **change** in **average mortality over age**

$$\Delta \bar{m}_j = \underbrace{\sum_k \frac{p_{jk} + p_{j+1,k}}{2} \Delta m_{jk}}_{\text{Direct change } \Delta \bar{m}_j^D} + \underbrace{\sum_k \frac{m_{jk} + m_{j+1,k}}{2} \Delta p_{jk}}_{\text{Compositional change } \Delta \bar{m}_j^C}$$

$$m_{jk} = \frac{D_{jk}}{E_{jk}}$$

$$p_{jk} = \frac{E_{jk}}{\sum_k E_{jk}}$$

$$\bar{m}_j = \frac{\sum_k D_{jk}}{\sum_k E_{jk}}$$

Data over models testing the selection hypothesis

Mortality selection along…
APGAR score ×
Birthweight ×
Gestation at delivery

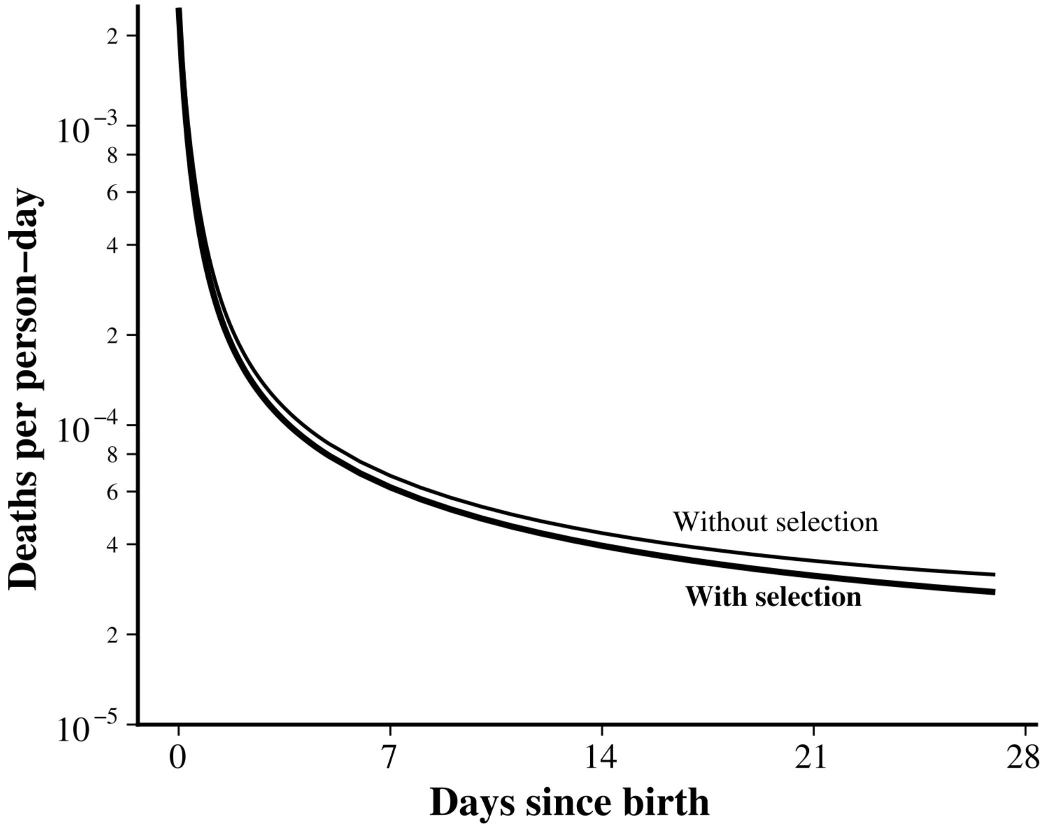
… explains
21% of the **mortality decline** over the first day of life,
and less than 5% at later ages

… explains
21% of the **decline in the variance of mortality risks** over the first day of life,
and less than 5% at later ages

… explains
23% of the **decline in the positive skewness of mortality risks** over the first day of life,
and less than 5% at later ages

Data: US infants born 2008-12. CDC/NCHS.

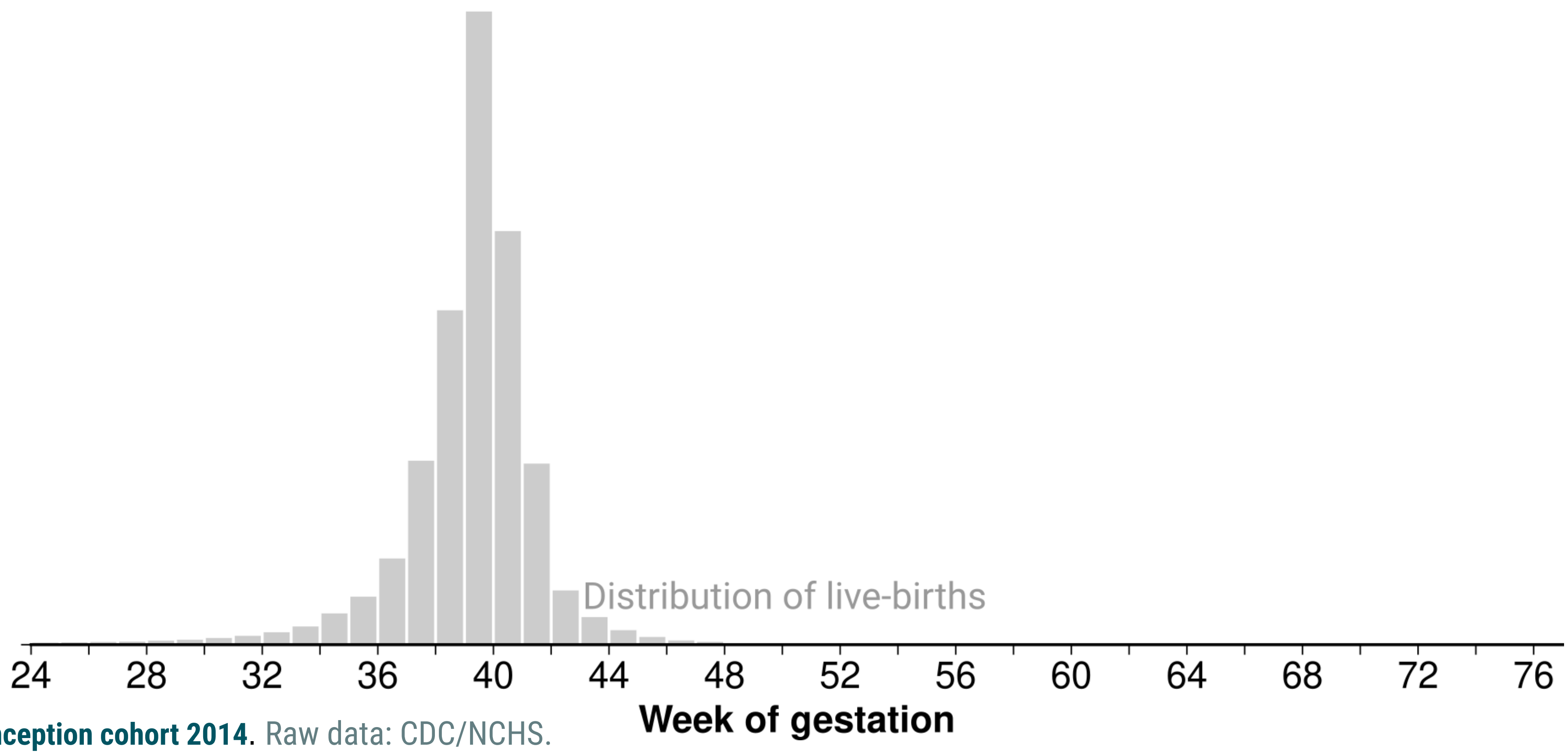
Data over models testing the selection hypothesis



Data: US infants born 2008-12. CDC/NCHS.

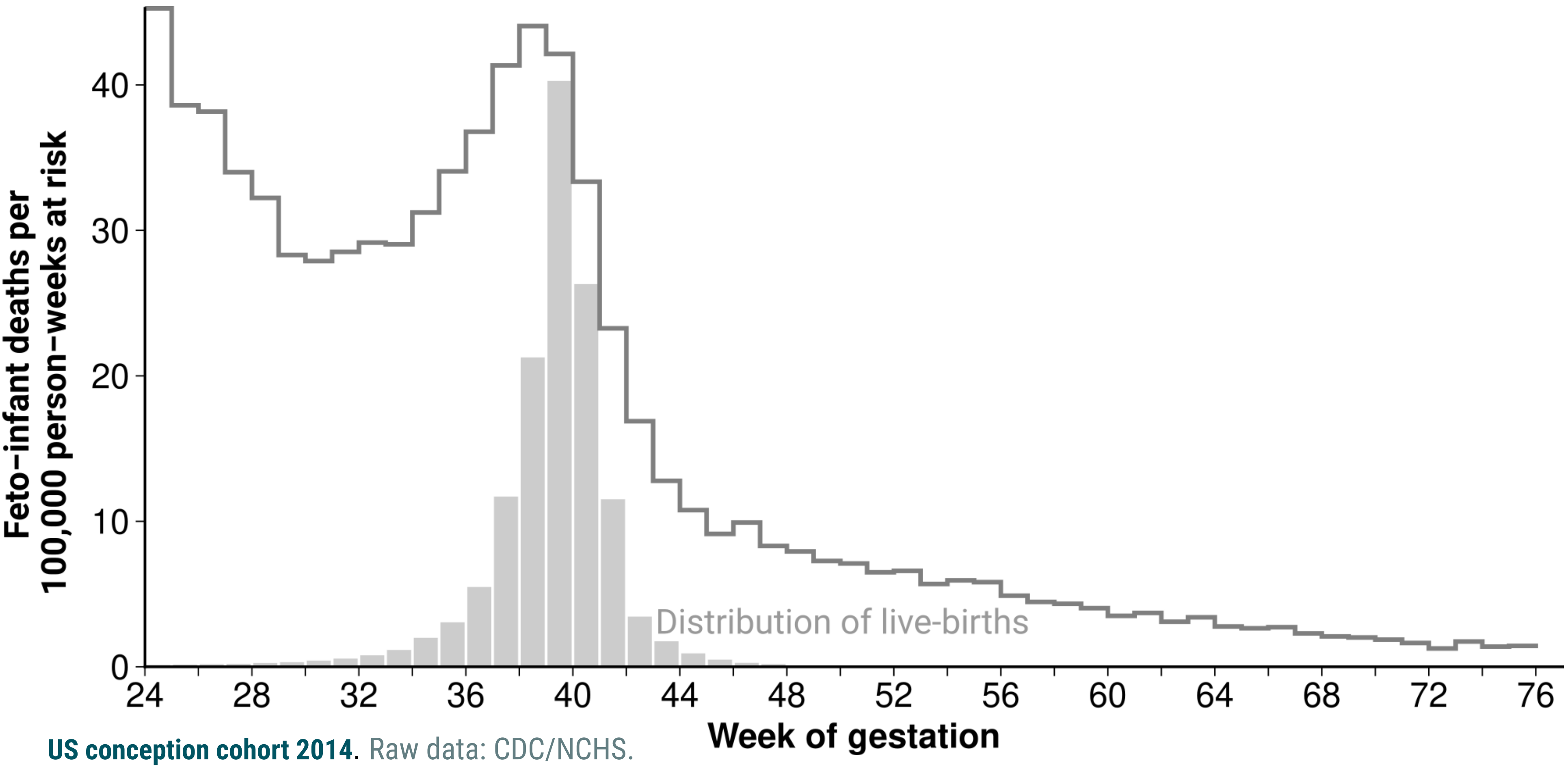
Model pluralism

Methodological pluralism Gompertz before birth



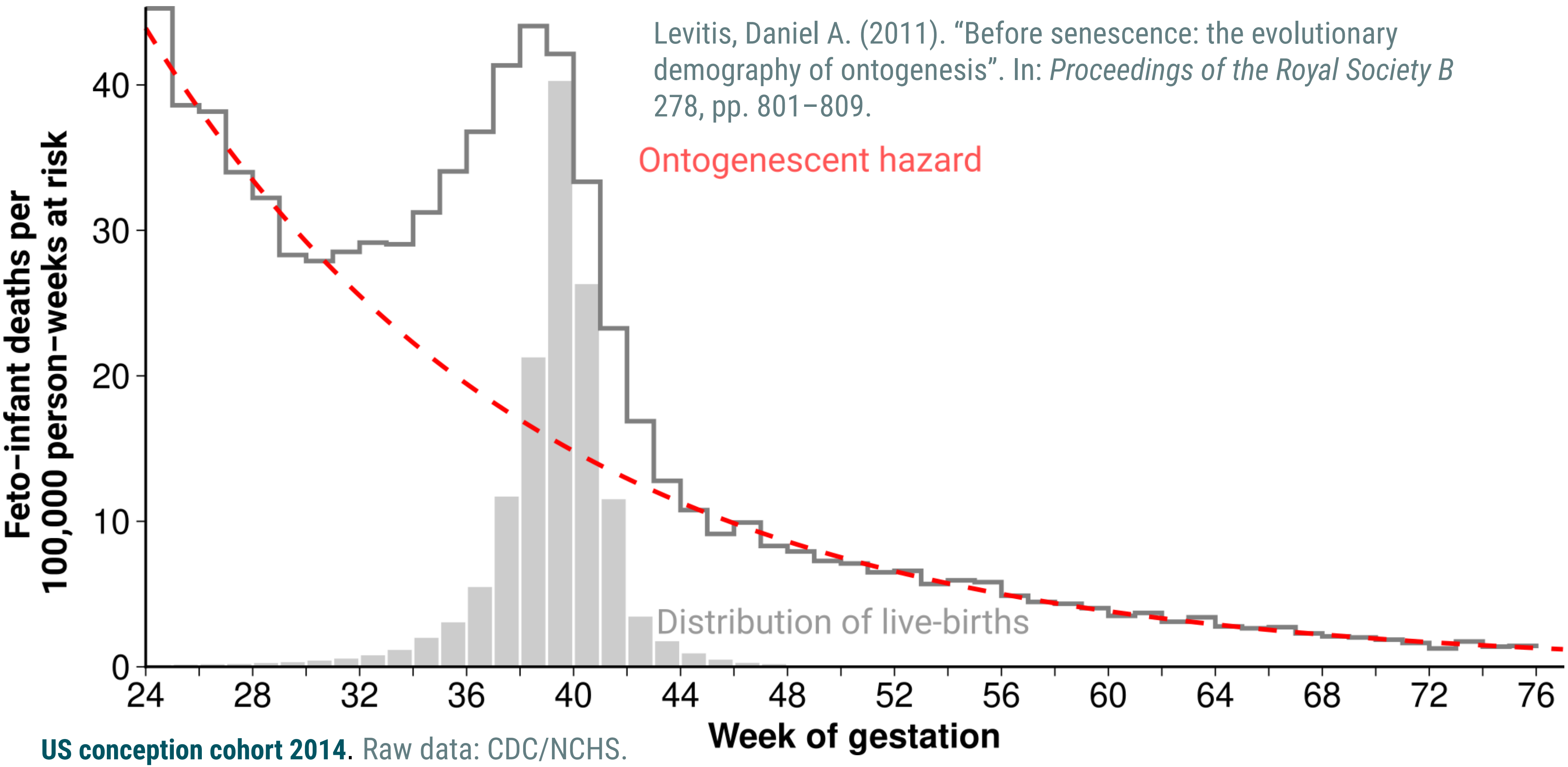
US conception cohort 2014. Raw data: CDC/NCHS.

Methodological pluralism Gompertz before birth



US conception cohort 2014. Raw data: CDC/NCHS.

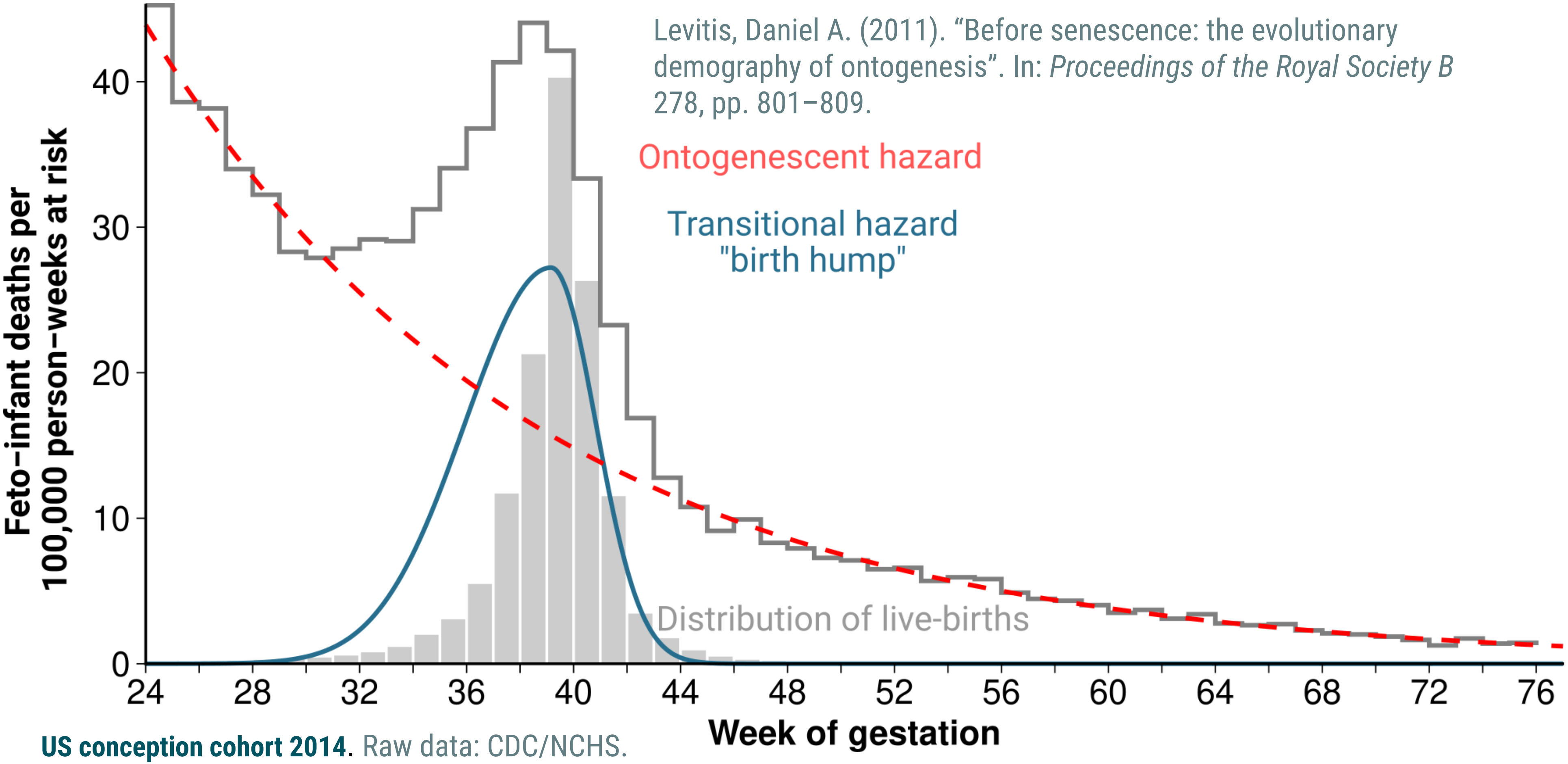
Methodological pluralism Gompertz before birth



Levitis, Daniel A. (2011). “Before senescence: the evolutionary demography of ontogenesis”. In: *Proceedings of the Royal Society B* 278, pp. 801–809.

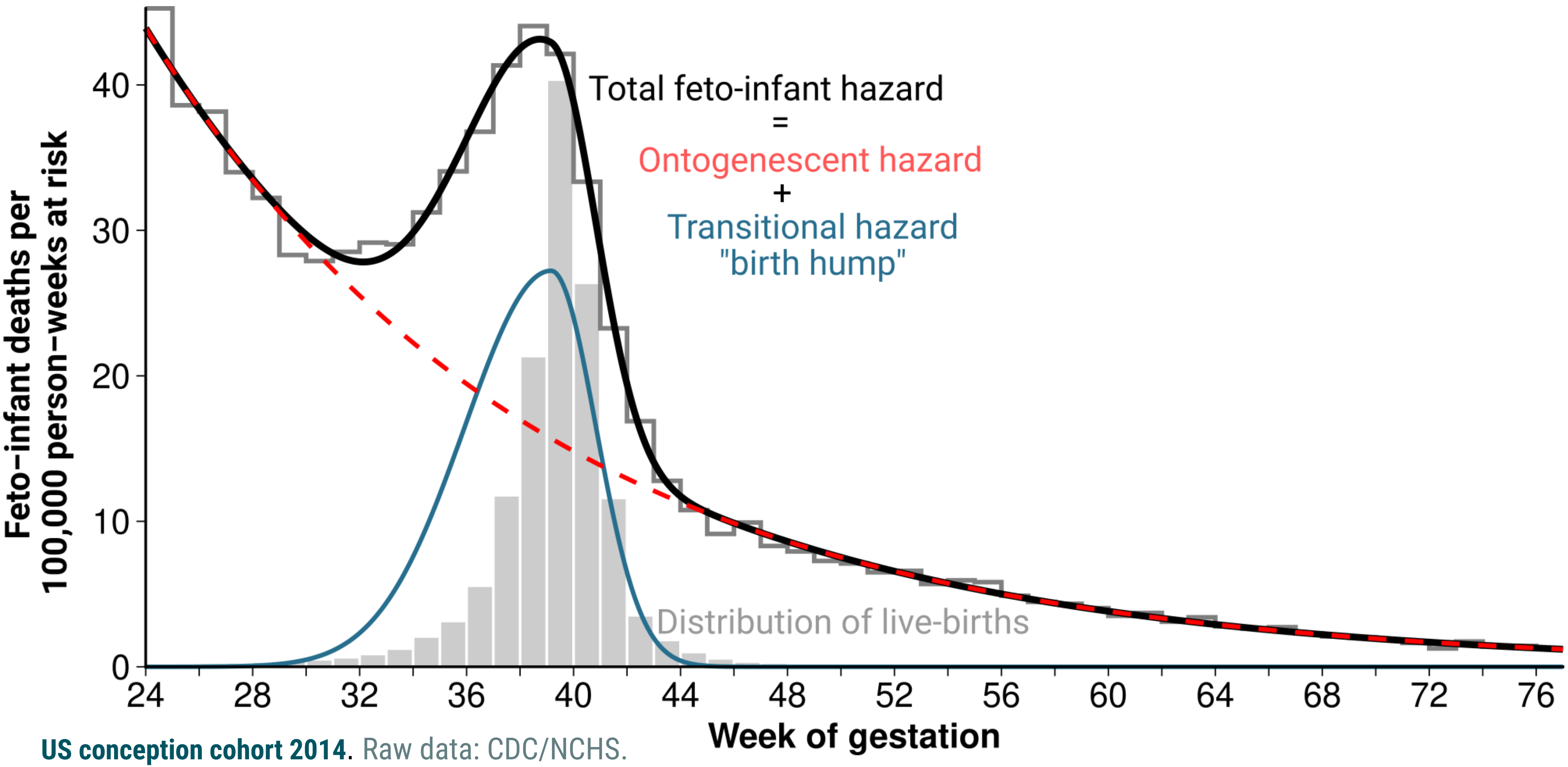
US conception cohort 2014. Raw data: CDC/NCHS.

Methodological pluralism Gompertz before birth



US conception cohort 2014. Raw data: CDC/NCHS.

Methodological pluralism Gompertz before birth



US conception cohort 2014. Raw data: CDC/NCHS.

Methodological pluralism Gompertz before birth

Combined feto-infant mortality by week of gestation among conception cohort 2014. The dynamics of ontogenescence.

Raw data: CDC/NCHS.

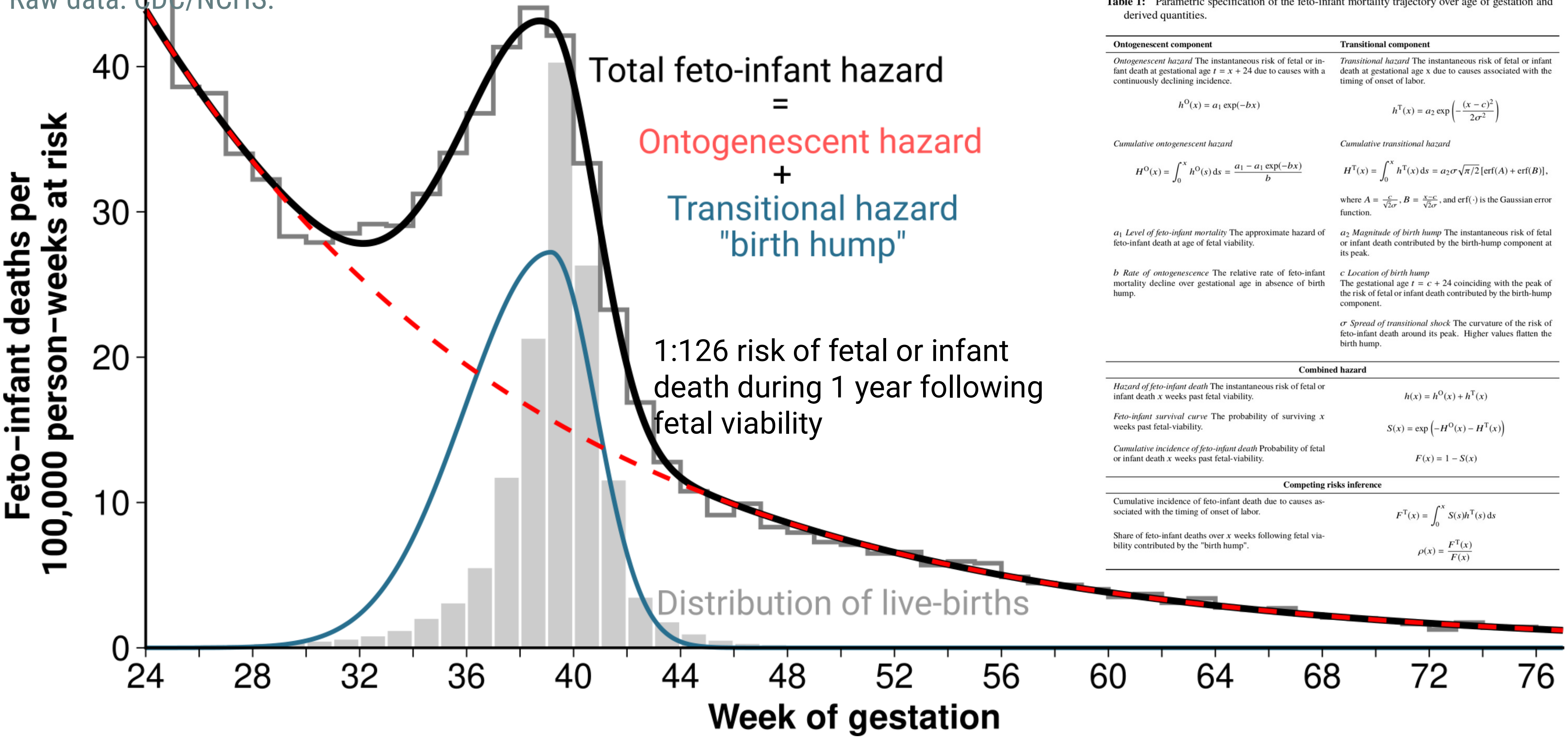
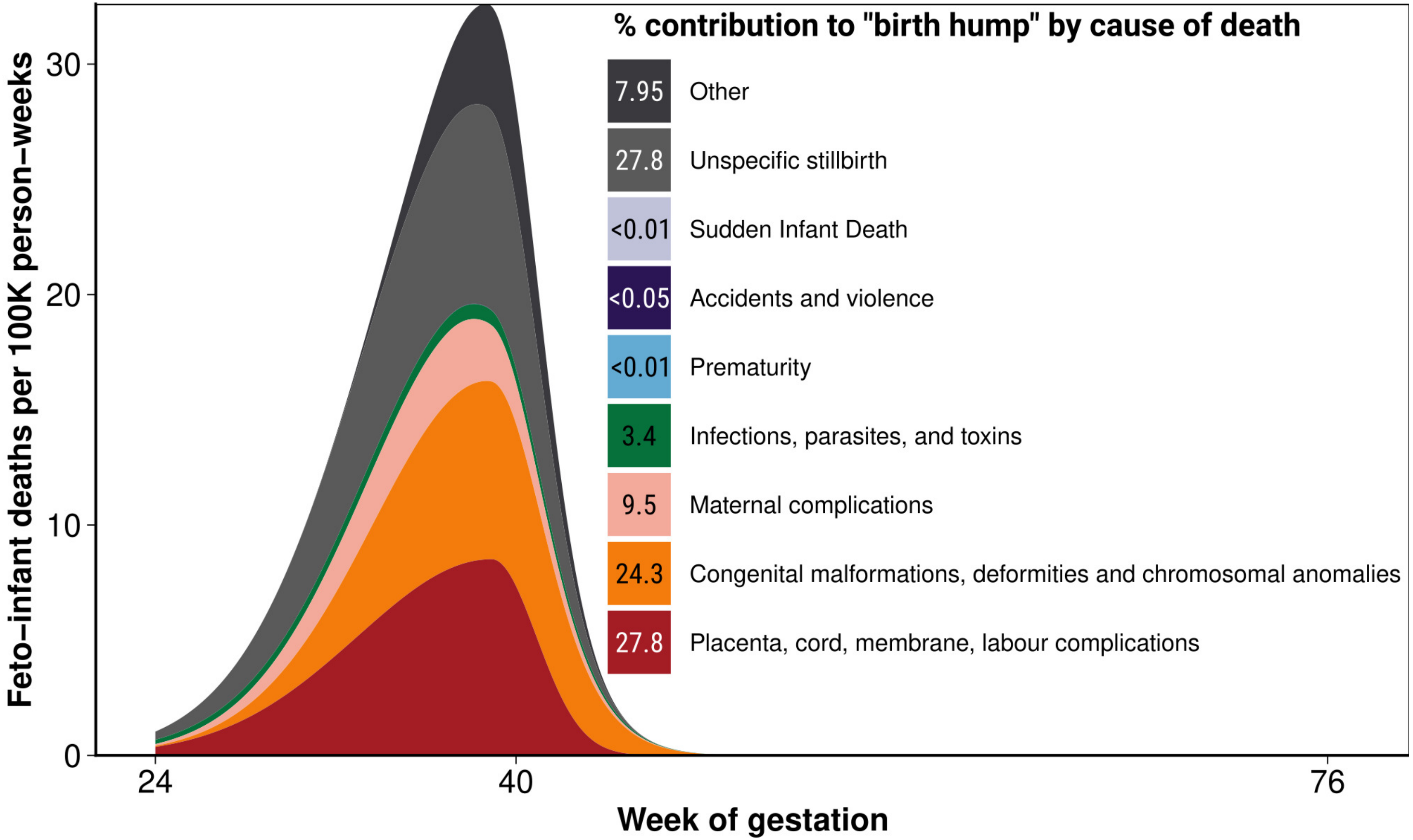


Table 1: Parametric specification of the feto-infant mortality trajectory over age of gestation and derived quantities.

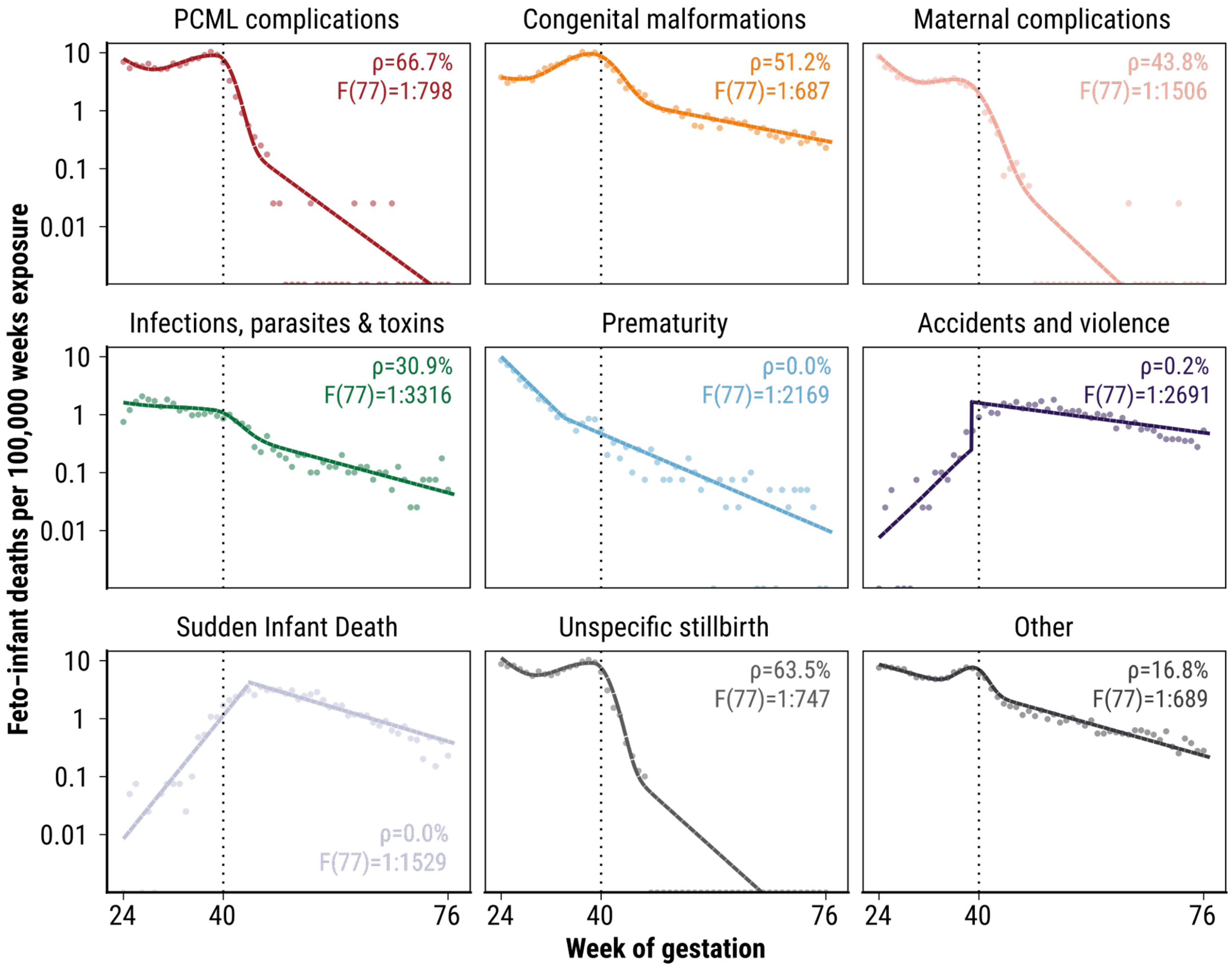
Ontogenescent component	**Transitional component**
Ontogenescent hazard The instantaneous risk of fetal or infant death at gestational age $t = x + 24$ due to causes with a continuously declining incidence.	*Transitional hazard* The instantaneous risk of fetal or infant death at gestational age x due to causes associated with the timing of onset of labor.
$h^{O}(x) = a_1 \exp(-bx)$	$h^{T}(x) = a_2 \exp\left(-\frac{(x-c)^2}{2\sigma^2}\right)$
Cumulative ontogenescent hazard	*Cumulative transitional hazard*
$H^{O}(x) = \int_0^x h^{O}(s)\,ds = \frac{a_1 - a_1 \exp(-bx)}{b}$	$H^{T}(x) = \int_0^x h^{T}(x)\,ds = a_2\sigma\sqrt{\pi/2}\,[\mathrm{erf}(A) + \mathrm{erf}(B)]$, where $A = \frac{c}{\sqrt{2}\sigma}$, $B = \frac{x-c}{\sqrt{2}\sigma}$, and $\mathrm{erf}(\cdot)$ is the Gaussian error function.
a_1 *Level of feto-infant mortality* The approximate hazard of feto-infant death at age of fetal viability.	a_2 *Magnitude of birth hump* The instantaneous risk of fetal or infant death contributed by the birth-hump component at its peak.
b *Rate of ontogenescence* The relative rate of feto-infant mortality decline over gestational age in absence of birth hump.	c *Location of birth hump* The gestational age $t = c + 24$ coinciding with the peak of the risk of fetal or infant death contributed by the birth-hump component.
	σ *Spread of transitional shock* The curvature of the risk of feto-infant death around its peak. Higher values flatten the birth hump.
Combined hazard	
Hazard of feto-infant death The instantaneous risk of fetal or infant death x weeks past fetal viability.	$h(x) = h^{O}(x) + h^{T}(x)$
Feto-infant survival curve The probability of surviving x weeks past fetal-viability.	$S(x) = \exp\left(-H^{O}(x) - H^{T}(x)\right)$
Cumulative incidence of feto-infant death Probability of fetal or infant death x weeks past fetal-viability.	$F(x) = 1 - S(x)$
Competing risks inference	
Cumulative incidence of feto-infant death due to causes associated with the timing of onset of labor.	$F^{T}(x) = \int_0^x S(s)h^{T}(s)\,ds$
Share of feto-infant deaths over x weeks following fetal viability contributed by the "birth hump".	$\rho(x) = \frac{F^{T}(x)}{F(x)}$

Methodological pluralism what can you *do* with a model

Cause of death decomposition of the "birth-hump" among US conception cohort 2014. Schöley & Kniffka (2025). "The birth-hump". A shape decomposition of perinatal excess mortality.

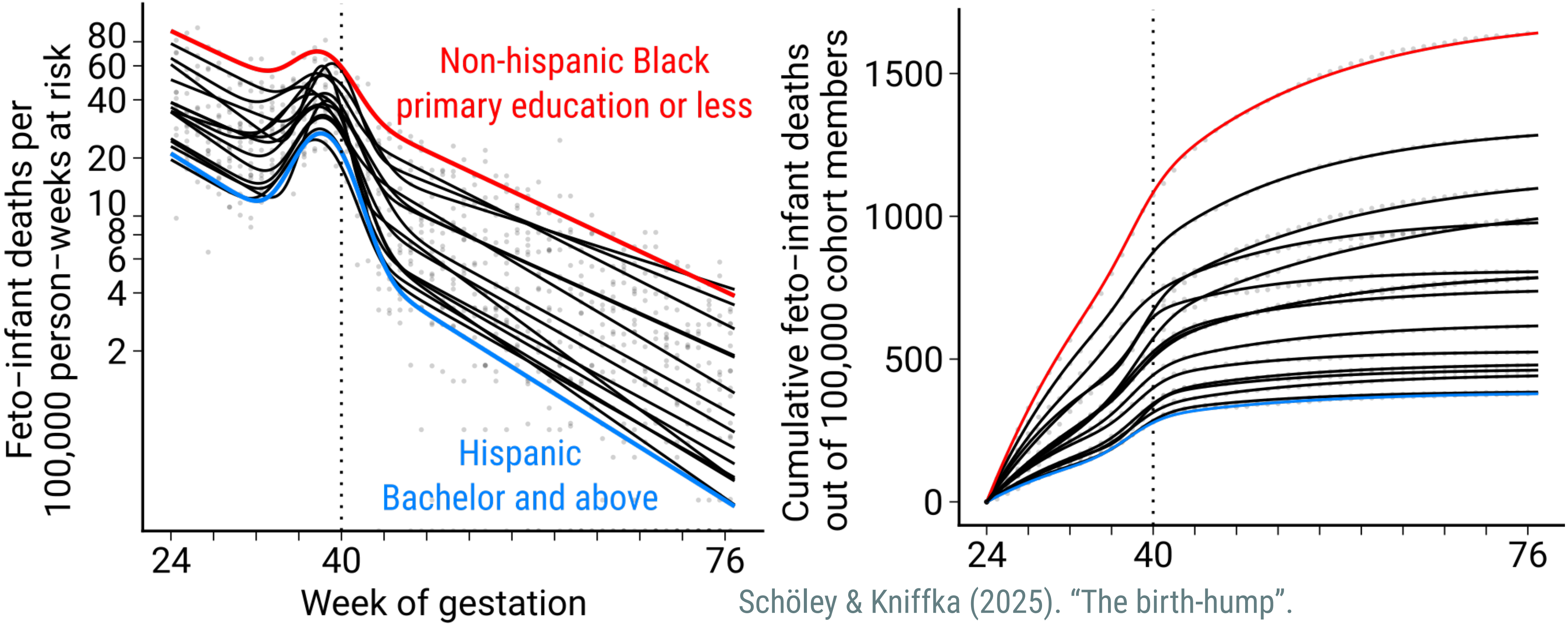


Methodological pluralism what can you *do* with a model



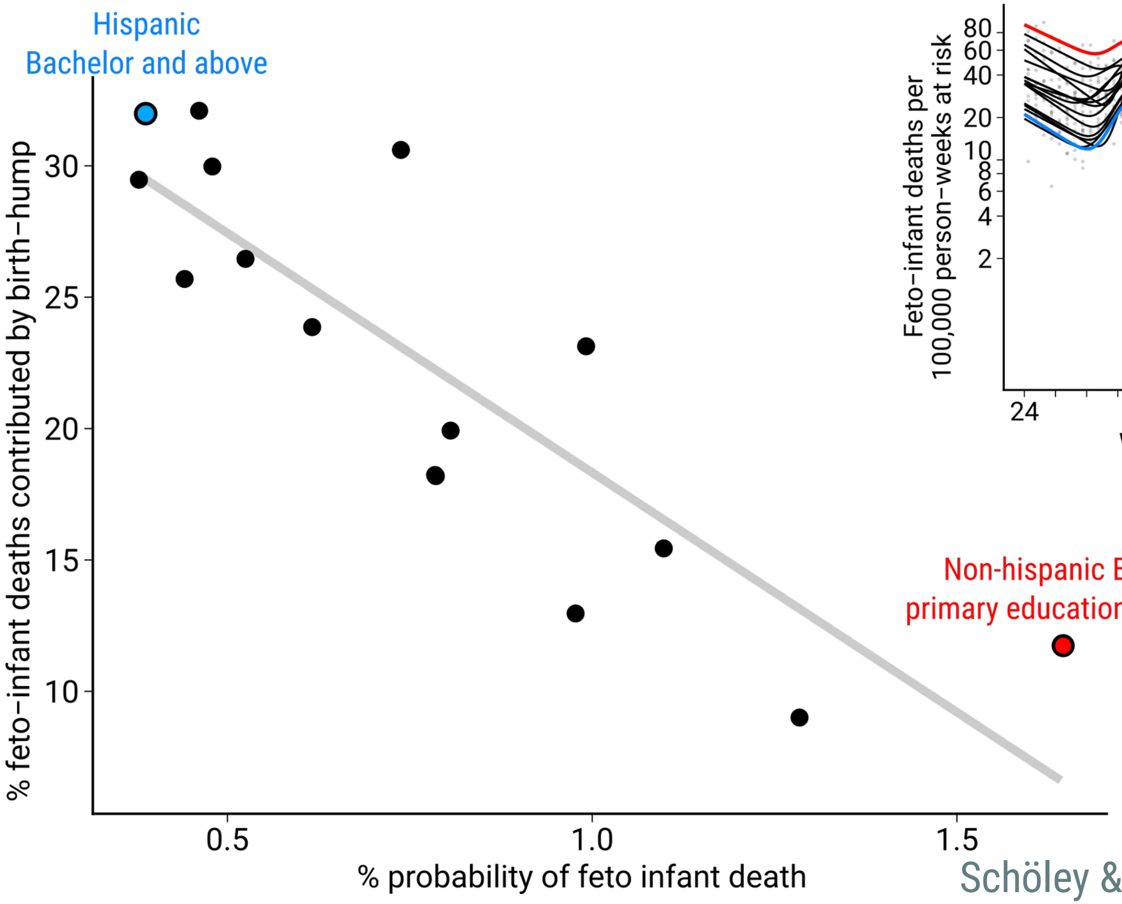
Schöley & Kniffka (2025). “The birth-hump”. A shape decomposition of perinatal excess mortality.

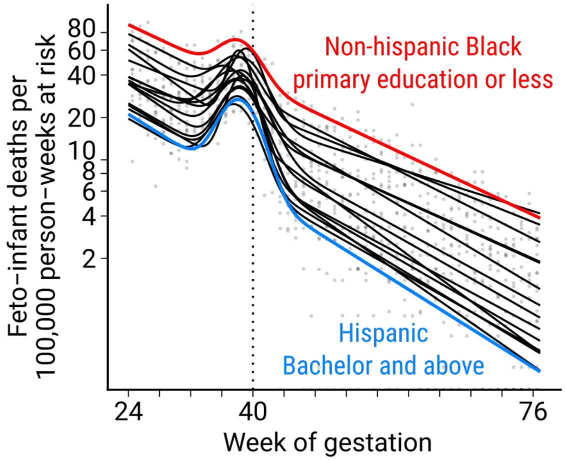
Methodological pluralism phenomenology + data + mechanisms

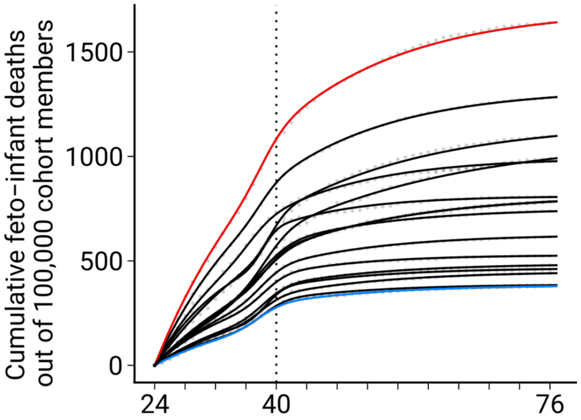


Schöley & Kniffka (2025). “The birth-hump”.
A shape decomposition of perinatal excess mortality.

Methodological pluralism phenomenology + mechanisms + data

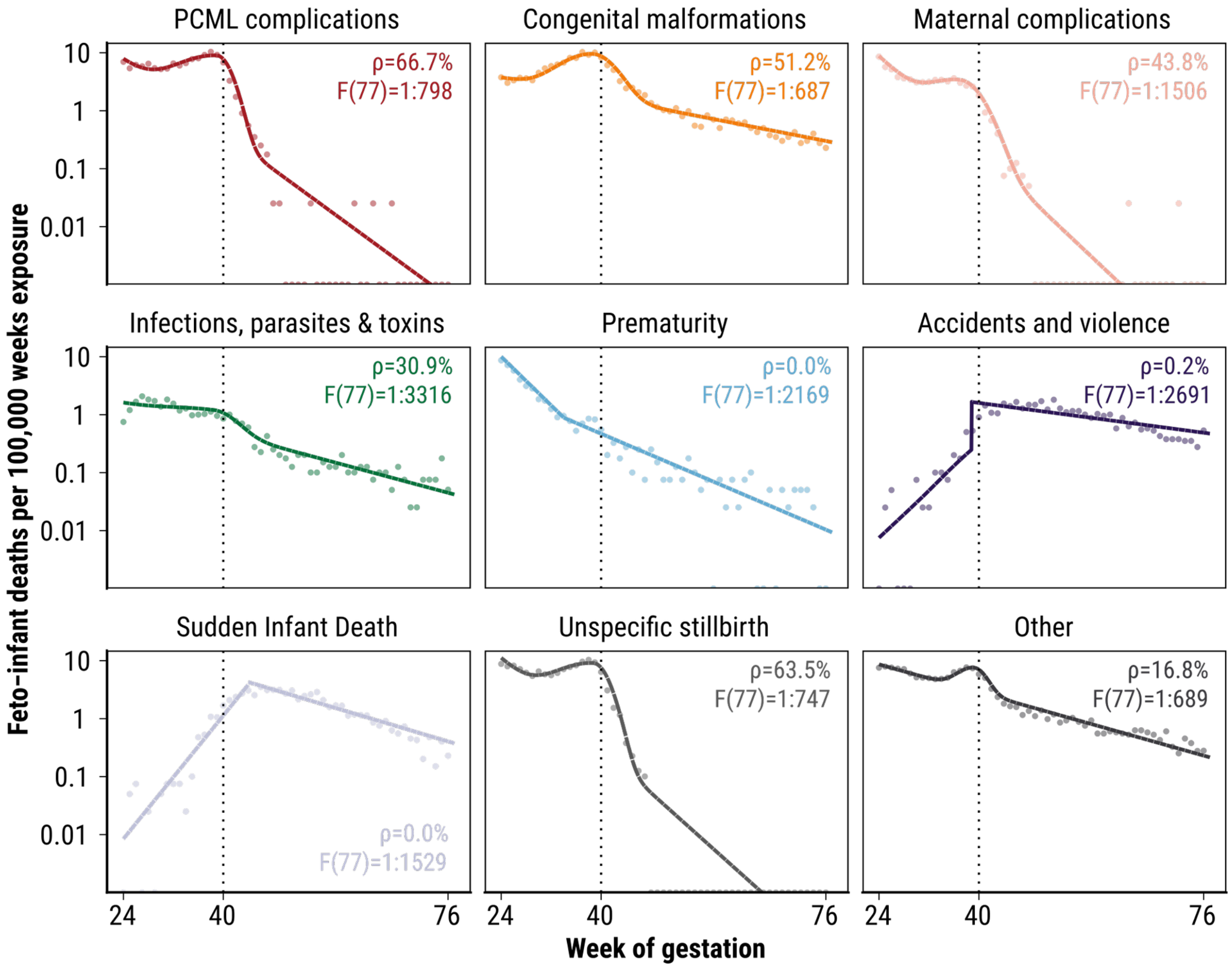






Schöley & Kniffka (2025). “The birth-hump”.
A shape decomposition of perinatal excess mortality.

Methodological pluralism all models are true, some describe reality



Schöley & Kniffka (2025). “The birth-hump”. A shape decomposition of perinatal excess mortality.

What remains?

Jonas Schöley
jschoeley.com

@jschoeley.com
0000-0002-3340-8518
schoeley@demogr.mpg.de

MAX PLANCK INSTITUTE
FOR DEMOGRAPHIC RESEARCH